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Analytic Bethe ansatz and functional relations related to tensor-like representations of type-II Lie superalgebras $B(r|s)$ and $D(r|s)$

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Abstract. An analytic Bethe ansatz is carried out related to tensor-like representations of the type-II Lie superalgebras $B(r|s) = \mathrm{osp}(2r+1|2s)$ ($r \in \mathbb{Z}_{\geqslant 0}$, $s \in \mathbb{Z}_{\geqslant 1}$) and $D(r|s) = \mathrm{osp}(2r|2s)$ ($r \in \mathbb{Z}_{\geqslant 2}$, $s \in \mathbb{Z}_{\geqslant 1}$). We present eigenvalue formulae of transfer matrices in dressed vacuum forms (DVFs) labelled by Young (super) diagrams. A class of transfer matrix functional relations (T-system) is discussed. In particular for the $B(0|s) = \mathrm{osp}(1|2s)$ ($s \in \mathbb{Z}_{\geqslant 1}$) case, a complete set of functional relations is proposed by using duality among DVFs.

1. Introduction

Solvable lattice models related to Lie superalgebras [1] have received much attention [2–9]. The construction of eigenvalue formulae of transfer matrices for such models is an important problem in mathematical physics. In order to achieve this, the Bethe ansatz has been often used.

Nowadays, there is much literature (see, for example, [3, 10–17] and references therein) on Bethe ansatz analysis for solvable lattice models related to Lie superalgebras. However, most of it deals only with models related to simple representations, like fundamental ones. Only a few people [13, 15, 17] have tried to deal with more complicated models such as fusion models [18] by Bethe ansatz, and there has been no systematic study of this subject.

To address this situation, we have recently executed [19–22] an analytic Bethe ansatz [23–29] systematically for the type-I Lie superalgebras $\mathrm{sl}(r+1|s+1)$, $C(s)$ cases. Namely, we have proposed a set of dressed vacuum forms (DVFs) and a class of functional relations for it. The purpose of this paper is to extend our recent work to type-II Lie superalgebras of the $B(r|s) = \mathrm{osp}(2r+1|2s)$ ($r \in \mathbb{Z}_{\geqslant 0}$, $s \in \mathbb{Z}_{\geqslant 1}$) and $D(r|s) = \mathrm{osp}(2r|2s)$ ($r \in \mathbb{Z}_{\geqslant 2}$, $s \in \mathbb{Z}_{\geqslant 1}$) cases.

We can express [3, 16] the Bethe ansatz equation (BAE) (3.1) by using the representation theoretical data of $B(r|s)$ ($r \in \mathbb{Z}_{\geqslant 1}$, $s \in \mathbb{Z}_{\geqslant 1}$) or $D(r|s)$ ($r \in \mathbb{Z}_{\geqslant 2}$, $s \in \mathbb{Z}_{\geqslant 1}$), as long as we adopt the distinguished simple root system [1]. On the other hand, $B(0|s) = \mathrm{osp}(1|2s)$ is a special case among the Lie superalgebras. In contrast to other Lie superalgebras, its simple root system is unique. Corresponding to this fact, BAEs (3.2)–(3.5) associated with the root system of $B(0|s)$ will be also unique. The peculiarity of these BAEs is that, so far, a naive description in terms of the simple root system does not exist for the sth BAEs (3.4) and (3.5), which correspond to the odd root α_s with $(\alpha_s|\alpha_s) \neq 0$ (cf [3, 16]).

We assume, as our starting point, the above-mentioned BAEs (3.1)–(3.5) for $B(r|s)$ and $D(r|s)$, and then carry out an analytic Bethe ansatz systematically to construct a class of DVFs. On constructing DVFs, *the pole-freeness under the BAE* and *the top-term hypothesis* [26, 27] play important roles.

We introduce skew-Young (super) diagrams $\lambda \subset \mu$, which are related to tensor-like representations† of $B(r|s)$ or $D(r|s)$. On these skew-Young (super) diagrams, we define a set of admissible tableaux $B(\lambda \subset \mu)$ with some semi-standard-like conditions. There is a one-to-one correspondence between these conditions for the $B(0|s)$ case and the conditions for the $A_{2s}^{(2)}$ case [29]. In addition, in contrast to the $B(r|s)$ case, these conditions for the $D(r|s)$ case have non-local nature. Next, we define a function $\mathcal{T}_{\lambda\subset\mu}(u)$ (3.38) of a spectral parameter u as summations over $B(\lambda \subset \mu)$. It will provide us with the spectra of a set of transfer matrices for various fusion $B(r|s)$ or $D(r|s)$ vertex models. It contains the top term [26, 27], which carries the highest weight of the irreducible representation of $B(r|s)$ or $D(r|s)$ labelled by a skew-Young (super) diagram $\lambda \subset \mu$. In particular, the simplest example of $\mathcal{T}_{\lambda\subset\mu}(u)$, that is $\mathcal{T}^1(u) = \mathcal{T}_1(u) = \mathcal{T}_{(1^1)}(u)$, reduces to the eigenvalue formula of the transfer matrix [16] of some vertex model related to the fundamental representation of $B(r|s)$ or $D(r|s)$ after some redefinitions. The BAEs (3.1)–(3.5) are assumed to be common to all the DVFs for transfer matrices with various fusion types in the auxiliary space as long as they act on a common quantum space. Therefore, we can prove the pole-freeness of $\mathcal{T}^a(u) = \mathcal{T}_{(1^a)}(u)$ for any $a \in \mathbb{Z}_{\geqslant 0}$ under the common BAEs (3.1)–(3.5). We further mention a determinant formula, by which $\mathcal{T}_{\lambda\subset\mu}(u)$ can be expressed only by the fundamental functions $\{\mathcal{T}^a\}$ and then pole-freeness follows immediately. A set of transfer matrix functional relations among DVFs also follows from this formula. It will be a kind of T-system [32] (see also [13,15,17,19–22,25,27,29,33–38]). In particular for the $B(0|s)$ case, there is remarkable duality among DVFs (see theorem 4.1 and (4.21)). On constructing the above-mentioned functional relations, this duality among DVFs plays an important role.

The outline of this paper is as follows. In section 2, we briefly mention the Lie superalgebras $B(r|s)$ and $D(r|s)$. In section 3, we execute an analytic Bethe ansatz based on the BAEs (3.1)–(3.5) associated with distinguished simple root systems. In section 4, we discuss transfer matrix functional relations. Section 5 is devoted to a summary and discussion. In appendices A.1–A.3, we prove the pole-freeness of DVFs. Appendix B provides a generating series of $\mathcal{T}^a(u)$ and $\mathcal{T}_m(u) = \mathcal{T}_{(m)}(u)$. In this paper, we adopt similar notation as in [19–22, 26, 27]. Finally, we note that we can recover many formulae in [26, 27] for B_r or D_r, if we set $s = 0$ and redefine the vacuum parts.

2. Lie superalgebras

In this section, we briefly mention the Lie superalgebras $B(r|s)$ and $D(r|s)$ (see, for example, [1, 39–43]).

There are several choices of simple root systems and the simplest one is the distinguished simple root system. They read as follows.

The $B(r|s)$ $(r, s \in \mathbb{Z}_{\geqslant 1})$ case (see figure 1):

$$\begin{aligned}
&\alpha_i = \delta_i - \delta_{i+1} \qquad i = 1, 2, \ldots, s-1\\
&\alpha_s = \delta_s - \epsilon_1\\
&\alpha_{s+j} = \epsilon_j - \epsilon_{j+1} \qquad j = 1, 2, \ldots, r-1\\
&\alpha_{s+r} = \epsilon_r.
\end{aligned} \tag{2.1}$$

† In this paper, we do not deal with spinorial representations.

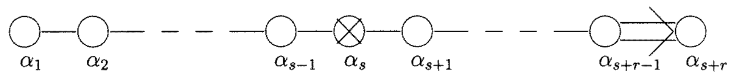

Figure 1. Dynkin diagram for the Lie superalgebra $B(r|s) = \mathrm{osp}(2r+1|2s)$ ($r \in \mathbb{Z}_{\geqslant 1}$, $s \in \mathbb{Z}_{\geqslant 1}$) corresponding to the distinguished simple root system: white circles denote even roots; the crossed circle denotes an odd root α with $(\alpha|\alpha) = 0$.

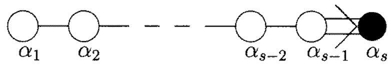

Figure 2. Dynkin diagram for the Lie superalgebra $B(0|s) = \mathrm{osp}(1|2s)$ ($s \in \mathbb{Z}_{\geqslant 1}$): the black circle denotes an odd root α with $(\alpha|\alpha) \neq 0$.

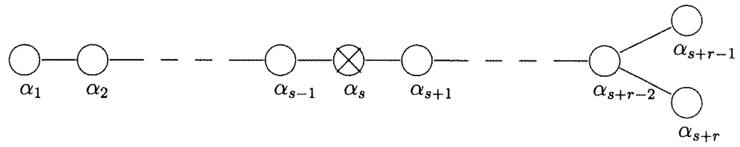

Figure 3. Dynkin diagram for the Lie superalgebra $D(r|s) = \mathrm{osp}(2r|2s)$ ($r \in \mathbb{Z}_{\geqslant 2}$, $s \in \mathbb{Z}_{\geqslant 1}$) corresponding to the distinguished simple root system.

The $B(0|s)$ ($s \in \mathbb{Z}_{\geqslant 1}$) case (see figure 2):

$$\begin{aligned} &\alpha_i = \delta_i - \delta_{i+1} \qquad \text{for} \quad i = 1, 2, \ldots, s-1 \\ &\alpha_s = \delta_s. \end{aligned} \tag{2.2}$$

The $D(r|s)$ ($r \in \mathbb{Z}_{\geqslant 2}$, $s \in \mathbb{Z}_{\geqslant 1}$) case (see figure 3):

$$\begin{aligned} &\alpha_i = \delta_i - \delta_{i+1} \qquad i = 1, 2, \ldots, s-1 \\ &\alpha_s = \delta_s - \epsilon_1 \\ &\alpha_{s+j} = \epsilon_j - \epsilon_{j+1} \qquad j = 1, 2, \ldots, r-2 \\ &\alpha_{s+r-1} = \epsilon_{r-1} - \epsilon_r \\ &\alpha_{s+r} = \epsilon_{r-1} + \epsilon_r \end{aligned} \tag{2.3}$$

where $\epsilon_1, \ldots, \epsilon_r; \delta_1, \ldots, \delta_s$ are the bases of the dual space of the Cartan subalgebra with the bilinear form (|) such that†

$$(\epsilon_i|\epsilon_j) = \delta_{ij} \qquad (\epsilon_i|\delta_j) = (\delta_i|\epsilon_j) = 0 \qquad (\delta_i|\delta_j) = -\delta_{ij}. \tag{2.4}$$

$\{\alpha_i\}_{i\neq s}$ are even roots and α_s is an odd root. Note that $(\alpha_s|\alpha_s) = 0$ for the $B(r|s)$ ($r, s \in \mathbb{Z}_{\geqslant 1}$) and $D(r|s)$ ($r \in \mathbb{Z}_{\geqslant 2}$, $s \in \mathbb{Z}_{\geqslant 1}$) cases, while $(\alpha_s|\alpha_s) \neq 0$ for the $B(0|s)$ ($s \in \mathbb{Z}_{\geqslant 1}$) case.

Let $\lambda \subset \mu$ be a skew-Young (super) diagram labelled by the sequences of non-negative integers $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$ such that $\mu_i \geqslant \lambda_i : i = 1, 2, \ldots;$ $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant 0$; $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant 0$ and $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$ be the conjugate of λ. In particular, for $B(r|s)$ ($r, s \in \mathbb{Z}_{\geqslant 1}$), the $\lambda = \phi$, $\mu_{r+1} \leqslant s$ case, the Kac–Dynkin label $[b_1, b_2, \ldots, b_{s+r}]$ is related to the Young (super) diagram with shape $\mu = (\mu_1, \mu_2, \ldots)$ as follows:

† We normalized the longest simple root as $|(\alpha|\alpha)| = 2$.

$$
\begin{aligned}
b_i &= \mu'_i - \mu'_{i+1} \qquad \text{for} \quad i \in \{1, 2, \ldots, s-1\} \\
b_s &= \mu'_s + \eta_1 \\
b_{s+j} &= \eta_j - \eta_{j+1} \qquad \text{for} \quad j \in \{1, 2, \ldots, r-1\} \\
b_{s+r} &= 2\eta_r
\end{aligned}
\tag{2.5}
$$

where $\eta_i = \mathrm{Max}\{\mu_i - s, 0\}$. For $B(0|s)$ ($s \in \mathbb{Z}_{\geqslant 1}$), $\lambda = \phi$ case, the Kac–Dynkin label $[b_1, b_2, \ldots, b_s]$ is related to the Young (super) diagram with shape $\mu = (\mu_1, \mu_2, \ldots)$ as follows:

$$
\begin{aligned}
b_i &= \mu'_i - \mu'_{i+1} \qquad \text{for} \quad i \in \{1, 2, \ldots, s-1\} \\
b_s &= 2\mu'_s.
\end{aligned}
\tag{2.6}
$$

For the $D(r|s)$ case, we use only the Young (super) diagram with shape $\mu = (1^a)$ or $\mu = (m^1)$. The Young (super) diagram with shape $\mu = (1^a)$ is related to the Kac–Dynkin label $[b_1, b_2, \ldots, b_{s+r}]$ as follows:

$$
b_j = a\delta_{j1}. \tag{2.7}
$$

And the Young (super) diagram with shape $\mu = (m^1)$ is related to the Kac–Dynkin label $[b_1, b_2, \ldots, b_{s+r}]$ as follows:

$$
b_j = \begin{cases} \delta_{jm} & \text{if} \quad m \in \{1, 2, \ldots, s\} \\ (m-s+1)\delta_{js} + (m-s)\delta_{js+1} & \text{if} \quad r \in \mathbb{Z}_{\geqslant 3} \quad m \in \mathbb{Z}_{\geqslant s+1} \\ (m-s+1)\delta_{js} + (m-s)(\delta_{js+1} + \delta_{js+2}) & \text{if} \quad r = 2 \quad m \in \mathbb{Z}_{\geqslant s+1}. \end{cases} \tag{2.8}
$$

An irreducible representation of $B(0|s)$ with the Kac–Dynkin label $[b_1, b_2, \ldots, b_s]$ is finite dimensional [39] if and only if

$$
\begin{aligned}
b_j &\in \mathbb{Z}_{\geqslant 0} \qquad \text{for} \quad j \in \{1, 2, \ldots, s-1\} \\
b_s &\in 2\mathbb{Z}_{\geqslant 0}.
\end{aligned}
\tag{2.9}
$$

The dimensionality of the irreducible representation $V[b_1, b_2, \ldots, b_s]$ of $B(0|s)$ with the highest weight labelled by the Kac–Dynkin label $[b_1, b_2, \ldots, b_s]$ is given [39, 43] as follows†:

$$
\begin{aligned}
\dim V[b_1, b_2, \ldots, b_s] = &\prod_{1\leqslant i<j\leqslant s} \frac{b_i + b_{i+1} + \cdots + b_{j-1} + j - i}{j - i} \\
&\times \frac{b_i + b_{i+1} + \cdots + b_{j-1} + 2(b_j + b_{j+1} + \cdots + b_{s-1}) + b_s + 2s - i - j + 1}{2s - i - j + 1} \\
&\times \prod_{1\leqslant k\leqslant s} \frac{2(b_k + b_{k+1} + \cdots + b_{s-1}) + b_s + 2s - 2k + 1}{2s - 2k + 1}.
\end{aligned}
\tag{2.10}
$$

3. Analytic Bethe ansatz

We assume, as our starting point, the following type of the Bethe ansatz equations‡ [3, 14, 27, 30, 31].

The $B(r|s)$ ($r, s \in \mathbb{Z}_{\geqslant 1}$) or $D(r|s)$ ($r \in \mathbb{Z}_{\geqslant 2}$, $s \in \mathbb{Z}_{\geqslant 1}$) case:

$$
-\left\{ \prod_{j=1}^{N} \frac{\Phi(u_k^{(a)} - w_j - 1)}{\Phi(u_k^{(a)} - w_j + 1)} \right\}^{\delta_{a1}} = (-1)^{\deg(\alpha_a)} \prod_{b=1}^{s+r} \frac{Q_b(u_k^{(a)} + (\alpha_a|\alpha_b))}{Q_b(u_k^{(a)} - (\alpha_a|\alpha_b))}. \tag{3.1}
$$

† We assume that $b_j + b_{j+1} + \cdots + b_{s-1} = 0$ if $j = s$.

‡ In this paper, we deal with the case, as an example, that the quantum spaces of the transfer matrices are fundamental representations.

The $B(0|s)$ $(s \in \mathbb{Z}_{\geqslant 2})$ case:

$$-\prod_{j=1}^{N} \frac{\Phi(u_k^{(1)} - w_j - 1)}{\Phi(u_k^{(1)} - w_j + 1)} = \frac{Q_1(u_k^{(1)} - 2) Q_2(u_k^{(1)} + 1)}{Q_1(u_k^{(1)} + 2) Q_2(u_k^{(1)} - 1)} \tag{3.2}$$

$$-1 = \frac{Q_{a-1}(u_k^{(a)} + 1) Q_a(u_k^{(a)} - 2) Q_{a+1}(u_k^{(a)} + 1)}{Q_{a-1}(u_k^{(a)} - 1) Q_a(u_k^{(a)} + 2) Q_{a+1}(u_k^{(a)} - 1)} \qquad \text{for} \quad 2 \leqslant a \leqslant s-1 \tag{3.3}$$

$$1 = \frac{Q_{s-1}(u_k^{(s)} + 1) Q_s(u_k^{(s)} + 1) Q_s(u_k^{(s)} - 2)}{Q_{s-1}(u_k^{(s)} - 1) Q_s(u_k^{(s)} - 1) Q_s(u_k^{(s)} + 2)}. \tag{3.4}$$

The $B(0|1)$ case:

$$\prod_{j=1}^{N} \frac{\Phi(u_k^{(1)} - w_j - 1)}{\Phi(u_k^{(1)} - w_j + 1)} = \frac{Q_1(u_k^{(1)} + 1) Q_1(u_k^{(1)} - 2)}{Q_1(u_k^{(1)} - 1) Q_1(u_k^{(1)} + 2)}. \tag{3.5}$$

Here $Q_a(u) = \prod_{j=1}^{N_a} \Phi(u - u_j^{(a)})$; $N \in \mathbb{Z}_{\geqslant 0}$ is the number of the lattice sites; $N_a \in \mathbb{Z}_{\geqslant 0}$; $u_j^{(a)}, w_j \in \mathbb{C}$; $a, k \in \mathbb{Z}$ $(a \in \{1, 2, \ldots, s+r\}$ $(r = 0$ for the $B(0|s)$ case); $k \in \{1, 2, \ldots, N_a\})$;

$$\deg(\alpha_a) = \begin{cases} 0 & \text{for even root} \\ 1 & \text{for odd root} \end{cases} = \delta_{a,s}. \tag{3.6}$$

Φ is a function, which has zero at $u = 0$. For example, $\Phi(u)$ has the following form:

$$\Phi(u) = u. \tag{3.7}$$

Remarkably, BAEs can be written in terms of root systems of Lie algebras [30, 31] or Lie superalgebras [3, 16]. Martins and Ramos [16] pointed out that $B(0|s)$ is an exception to this observation (see also [3]). To put it more precisely, an exception lies in the right-hand side of (3.4) and (3.5), which correspond to the odd root α_s with $(\alpha_s|\alpha_s) \neq 0$. In fact, one can derive (3.2) and (3.3) from (3.1) and (2.2); while one cannot derive (3.4) and (3.5).

Remark. There are compact expressions of BAEs for twisted quantum affine algebras [30]. Moreover, the BAEs (3.2)–(3.5) resemble the BAEs for $A_{2s}^{(2)}$. This resemblance will originate from the resemblance between $B(0|s)^{(1)}$ and $A_{2s}^{(2)}$. Thus there is a possibility that the BAEs (3.2)–(3.5) are also compactly written in terms of the root system of the Lie superalgebra $B(0|s)$ $(s \in \mathbb{Z}_{\geqslant 1})$. We also point out that the expression (3.1) is not always valid for non-distinguished simple root systems. In fact, we have confirmed for several cases that the Bethe ansatz equations corresponding to the odd roots α with $(\alpha|\alpha) \neq 0$ have a similar structure to (3.4) or (3.5) by using the correspondence [20] between the particle–hole transformation and the (super) Weyl reflection.

We define the set

$$J = J_+ \cup J_- \tag{3.8}$$

where

$$J_- = \{1, 2, \ldots, s, \overline{s}, \ldots, \overline{2}, \overline{1}\} \tag{3.9}$$

is common for $B(r|s)$ and $D(r|s)$; while J_+ is not:

$$J_+ = \{s+1, s+2, \ldots, s+r, \overline{s+r}, \ldots, \overline{s+2}, \overline{s+1}\} \cup \{0\} \qquad \text{for} \quad B(r|s)$$

$$J_+ = \{s+1, s+2, \ldots, s+r, \overline{s+r}, \ldots, \overline{s+2}, \overline{s+1}\} \qquad \text{for} \quad D(r|s).$$

On this set J, we define the total order

$$1 \prec 2 \prec \cdots \prec s+r \prec 0 \prec \overline{s+r} \prec \cdots \prec \overline{2} \prec \overline{1} \tag{3.10}$$

for the $B(r|s)$ case, and the partial order

$$1 \prec 2 \prec \cdots \prec s+r-1 \prec \begin{matrix} s+r \\ \\ \overline{s+r} \end{matrix} \prec \overline{s+r-1} \prec \cdots \prec \overline{2} \prec \overline{1} \tag{3.11}$$

for the $D(r|s)$ case. In contrast to the $B(r|s)$ case, there is no order between $s+r$ and $\overline{s+r}$ for the $D(r|s)$ case. We also define the grading parameter as follows:

$$p(a) = \begin{cases} 0 & \text{for} \quad a \in J_+ \\ 1 & \text{for} \quad a \in J_- \end{cases}. \tag{3.12}$$

For $a \in J$, we define† the following functions.

The $B(r|s)$ $(r \in \mathbb{Z}_{\geqslant 0}, s \in \mathbb{Z}_{\geqslant 1})$ case:

$$\begin{aligned}
\boxed{a}_u &= \psi_a(u)\frac{Q_{a-1}(u-a-1)Q_a(u-a+2)}{Q_{a-1}(u-a+1)Q_a(u-a)} \qquad \text{for} \quad 1 \leqslant a \leqslant s \\
\boxed{a}_u &= \psi_a(u)\frac{Q_{a-1}(u-2s+a+1)Q_a(u-2s+a-2)}{Q_{a-1}(u-2s+a-1)Q_a(u-2s+a)} \qquad \text{for} \quad s+1 \leqslant a \leqslant s+r \\
\boxed{0}_u &= \psi_0(u)\frac{Q_{s+r}(u-s+r+1)Q_{s+r}(u-s+r-2)}{Q_{s+r}(u-s+r-1)Q_{s+r}(u-s+r)} \\
\boxed{\overline{a}}_u &= \psi_{\overline{a}}(u)\frac{Q_{a-1}(u+2r-a-2)Q_a(u+2r-a+1)}{Q_{a-1}(u+2r-a)Q_a(u+2r-a-1)} \qquad \text{for} \quad s+1 \leqslant a \leqslant s+r \\
\boxed{\overline{a}}_u &= \psi_{\overline{a}}(u)\frac{Q_{a-1}(u-2s+2r+a)Q_a(u-2s+2r+a-3)}{Q_{a-1}(u-2s+2r+a-2)Q_a(u-2s+2r+a-1)} \qquad \text{for} \quad 1 \leqslant a \leqslant s.
\end{aligned} \tag{3.13}$$

The $D(r|s)$ $(r \in \mathbb{Z}_{\geqslant 2}, s \in \mathbb{Z}_{\geqslant 1})$ case:

$$\begin{aligned}
\boxed{a}_u &= \psi_a(u)\frac{Q_{a-1}(u-a-1)Q_a(u-a+2)}{Q_{a-1}(u-a+1)Q_a(u-a)} \qquad \text{for} \quad 1 \leqslant a \leqslant s \\
\boxed{a}_u &= \psi_a(u)\frac{Q_{a-1}(u-2s+a+1)Q_a(u-2s+a-2)}{Q_{a-1}(u-2s+a-1)Q_a(u-2s+a)} \\
&\quad \text{for} \quad s+1 \leqslant a \leqslant s+r-2 \\
\boxed{r+s-1}_u &= \psi_{r+s-1}(u)\frac{Q_{s+r-2}(u-s+r)Q_{s+r-1}(u-s+r-3)}{Q_{s+r-2}(u-s+r-2)Q_{s+r-1}(u-s+r-1)} \\
&\quad \times \frac{Q_{s+r}(u-s+r-3)}{Q_{s+r}(u-s+r-1)} \\
\boxed{r+s}_u &= \psi_{r+s}(u)\frac{Q_{s+r-1}(u-s+r+1)Q_{s+r}(u-s+r-3)}{Q_{s+r-1}(u-s+r-1)Q_{s+r}(u-s+r-1)} \\
\boxed{\overline{r+s}}_u &= \psi_{\overline{r+s}}(u)\frac{Q_{s+r-1}(u-s+r-3)Q_{s+r}(u-s+r+1)}{Q_{s+r-1}(u-s+r-1)Q_{s+r}(u-s+r-1)} \\
\boxed{\overline{r+s-1}}_u &= \psi_{\overline{r+s-1}}(u)\frac{Q_{s+r-2}(u-s+r-2)Q_{s+r-1}(u-s+r+1)}{Q_{s+r-2}(u-s+r)Q_{s+r-1}(u-s+r-1)} \\
&\quad \times \frac{Q_{s+r}(u-s+r+1)}{Q_{s+r}(u-s+r-1)} \\
\boxed{\overline{a}}_u &= \psi_{\overline{a}}(u)\frac{Q_{a-1}(u+2r-a-3)Q_a(u+2r-a)}{Q_{a-1}(u+2r-a-1)Q_a(u+2r-a-2)} \\
&\quad \text{for} \quad s+1 \leqslant a \leqslant s+r-2 \\
\boxed{\overline{a}}_u &= \psi_{\overline{a}}(u)\frac{Q_{a-1}(u-2s+2r+a-1)Q_a(u-2s+2r+a-4)}{Q_{a-1}(u-2s+2r+a-3)Q_a(u-2s+2r+a-2)} \\
&\quad \text{for} \quad 1 \leqslant a \leqslant s.
\end{aligned} \tag{3.14}$$

† In this paper, we often abbreviate the spectral parameter u.

Here we assume $Q_0(u) = 1$. The vacuum parts of the functions $\boxed{\mathrm{a}}_u$ (3.13) and (3.14) are given as follows. For the $B(r|s)$ ($r \in \mathbb{Z}_{\geqslant 0}, s \in \mathbb{Z}_{\geqslant 1}$) case:

$$\begin{aligned} &\psi_1(u) = \phi(u-2)\phi(u-2s+2r-1) \\ &\psi_a(u) = \phi(u)\phi(u-2s+2r-1) \qquad \text{for} \quad 2 \preceq a \preceq \overline{2} \\ &\psi_{\overline{1}}(u) = \phi(u)\phi(u-2s+2r+1). \end{aligned} \tag{3.15}$$

For the $D(r|s)$ ($r \in \mathbb{Z}_{\geqslant 2}, s \in \mathbb{Z}_{\geqslant 1}$) case:

$$\begin{aligned} &\psi_1(u) = \phi(u-2)\phi(u-2s+2r-2) \\ &\psi_a(u) = \phi(u)\phi(u-2s+2r-2) \qquad \text{for} \quad 2 \preceq a \preceq \overline{2} \\ &\psi_{\overline{1}}(u) = \phi(u)\phi(u-2s+2r). \end{aligned} \tag{3.16}$$

Here

$$\phi(u) = \prod_{j=1}^{N} \Phi(u - w_j). \tag{3.17}$$

Under the BAEs (3.1)–(3.5), we have† the following.

The $B(r|s)$ ($r, s \in \mathbb{Z}_{\geqslant 1}$) case:

$$\mathrm{Res}_{u=d+u_k^{(d)}}(\boxed{d}_u + \boxed{d+1}_u) = 0 \qquad \text{for} \quad 1 \leqslant d \leqslant s-1 \tag{3.18}$$

$$\mathrm{Res}_{u=s+u_k^{(s)}}(\boxed{s}_u - \boxed{s+1}_u) = 0 \tag{3.19}$$

$$\mathrm{Res}_{u=2s-d+u_k^{(d)}}(\boxed{d}_u + \boxed{d+1}_u) = 0 \qquad \text{for} \quad s+1 \leqslant d \leqslant s+r-1 \tag{3.20}$$

$$\mathrm{Res}_{u=s-r+u_k^{(s+r)}}(\boxed{s+r}_u + \boxed{0}_u) = 0 \tag{3.21}$$

$$\mathrm{Res}_{u=s-r+1+u_k^{(s+r)}}(\boxed{0}_u + \boxed{\overline{s+r}}_u) = 0 \tag{3.22}$$

$$\mathrm{Res}_{u=d-2r+1+u_k^{(d)}}(\boxed{\overline{d+1}}_u + \boxed{\overline{d}}_u) = 0 \qquad \text{for} \quad s+1 \leqslant d \leqslant s+r-1 \tag{3.23}$$

$$\mathrm{Res}_{u=s-2r+1+u_k^{(s)}}(\boxed{\overline{s+1}}_u - \boxed{\overline{s}}_u) = 0 \tag{3.24}$$

$$\mathrm{Res}_{u=-d+2s-2r+1+u_k^{(d)}}(\boxed{\overline{d+1}}_u + \boxed{\overline{d}}_u) = 0 \qquad \text{for} \quad 1 \leqslant d \leqslant s-1. \tag{3.25}$$

The $B(0|s)$ ($s \in \mathbb{Z}_{\geqslant 1}$) case:

$$\mathrm{Res}_{u=d+u_k^{(d)}}(\boxed{d}_u + \boxed{d+1}_u) = 0 \qquad \text{for} \quad 1 \leqslant d \leqslant s-1 \tag{3.26}$$

$$\mathrm{Res}_{u=s+u_k^{(s)}}(\boxed{s}_u - \boxed{0}_u) = 0 \tag{3.27}$$

$$\mathrm{Res}_{u=s+1+u_k^{(s)}}(\boxed{0}_u - \boxed{\overline{s}}_u) = 0 \tag{3.28}$$

$$\mathrm{Res}_{u=-d+2s+1+u_k^{(d)}}(\boxed{\overline{d+1}}_u + \boxed{\overline{d}}_u) = 0 \qquad \text{for} \quad 1 \leqslant d \leqslant s-1. \tag{3.29}$$

The $D(r|s)$ ($r \in \mathbb{Z}_{\geqslant 2}, s \in \mathbb{Z}_{\geqslant 1}$) case:

$$\mathrm{Res}_{u=d+u_k^{(d)}}(\boxed{d}_u + \boxed{d+1}_u) = 0 \qquad \text{for} \quad 1 \leqslant d \leqslant s-1 \tag{3.30}$$

$$\mathrm{Res}_{u=s+u_k^{(s)}}(\boxed{s}_u - \boxed{s+1}_u) = 0 \tag{3.31}$$

$$\mathrm{Res}_{u=2s-d+u_k^{(d)}}(\boxed{d}_u + \boxed{d+1}_u) = 0 \qquad \text{for} \quad s+1 \leqslant d \leqslant s+r-1 \tag{3.32}$$

$$\mathrm{Res}_{u=s-r+1+u_k^{(s+r)}}(\boxed{s+r-1}_u + \boxed{\overline{s+r}}_u) = 0 \tag{3.33}$$

† Here $\mathrm{Res}_{u=a} f(u)$ denotes the residue of a function $f(u)$ at $u = a$.

$$\mathrm{Res}_{u=s-r+1+u_k^{(s+r)}}(\boxed{s+r}_u + \boxed{\overline{s+r-1}}_u) = 0 \tag{3.34}$$

$$\mathrm{Res}_{u=d-2r+2+u_k^{(d)}}(\boxed{\overline{d+1}}_u + \boxed{\overline{d}}_u) = 0 \qquad \text{for} \quad s+1 \leqslant d \leqslant s+r-1 \tag{3.35}$$

$$\mathrm{Res}_{u=s-2r+2+u_k^{(s)}}(\boxed{\overline{s+1}}_u - \boxed{\overline{s}}_u) = 0 \tag{3.36}$$

$$\mathrm{Res}_{u=-d+2s-2r+2+u_k^{(d)}}(\boxed{\overline{d+1}}_u + \boxed{\overline{d}}_u) = 0 \qquad \text{for} \quad 1 \leqslant d \leqslant s-1. \tag{3.37}$$

We assign coordinates $(i, j) \in \mathbb{Z}^2$ on the skew-Young superdiagram $\lambda \subset \mu$ such that the row index i increases as we go downwards and the column index j increases as we go from the left to the right and that $(1, 1)$ is on the top left corner of μ. We define an admissible tableau b on the skew-Young superdiagram $\lambda \subset \mu$ as a set of elements $b(i, j) \in J$ labelled by the coordinates (i, j) mentioned above, with the following rule. The admissible condition for $B(r|s)$ $(r \in \mathbb{Z}_{\geqslant 0}, s \in \mathbb{Z}_{\geqslant 1})$:

(1)

$$b(i, j) \preceq b(i, j+1)$$

(2)

$$b(i, j) \preceq b(i+1, j)$$

(3)

$$b(i, j) \prec b(i+1, j) \qquad \text{if} \quad b(i, j) \in J_+ \setminus \{0\}$$

(4)

$$b(i, j) \prec b(i, j+1) \qquad \text{if} \quad b(i, j) \in J_- \cup \{0\}.$$

The admissible condition for $D(r|s)$ $(r \in \mathbb{Z}_{\geqslant 2}, s \in \mathbb{Z}_{\geqslant 1})$; $\lambda = \phi$; $\mu = (1^a)$:

(1)

$$b(k, 1) \preceq b(k+1, 1) \qquad \text{if} \quad b(k+1, 1) \in J_-$$

(2)

$$b(k, 1) \prec b(k+1, 1) \qquad \text{if} \quad b(k+1, 1) \in J_+$$

unless

$$(b(k, 1), b(k+1, 1)) = (\overline{s+r}, s+r) \qquad \text{or} \quad (s+r, \overline{s+r}).$$

The admissible condition for $D(r|s)$ $(r \in \mathbb{Z}_{\geqslant 2}, s \in \mathbb{Z}_{\geqslant 1})$; $\lambda = \phi$; $\mu = (m^1)$:

(1)

$$b(1, k) \preceq b(1, k+1) \qquad \text{if} \quad b(1, k+1) \in J_+$$

(2)

$$b(1, k) \prec b(1, k+1) \qquad \text{if} \quad b(1, k+1) \in J_-$$

(3) $s+r$ and $\overline{s+r}$ do not appear simultaneously.

Let $B(\lambda \subset \mu)$ be the set of admissible tableaux† on $\lambda \subset \mu$. We shall present a function $\mathcal{T}_{\lambda\subset\mu}(u)$ with a spectral parameter $u \in \mathrm{C}$ and skew-Young superdiagrams $\lambda \subset \mu$, which is a candidate of a set of DVFs for various fusion types in the auxiliary spaces‡ of transfer matrices of $B(r|s)$ or $D(r|s)$ vertex models. For the skew-Young (super) diagrams $\lambda \subset \mu$, define $\mathcal{T}_{\lambda\subset\mu}(u)$ as follows:

$$\mathcal{T}_{\lambda\subset\mu}(u) = \sum_{b\in B(\lambda\subset\mu)} \prod_{(i,j)\in(\lambda\subset\mu)} (-1)^{p(b(i,j))} \boxed{b(i,j)}_{u-\mu_1+\mu_1'-2i+2j} \tag{3.38}$$

where the product is taken over the coordinates (i, j) on $\lambda \subset \mu$.

We can express $\mathcal{T}_{\lambda\subset\mu}(u)$ as determinants over matrices, whose matrix elements are $\mathcal{T}^a$ or $\mathcal{T}_m$§ (cf [25, 27]). For the $B(r|s)$ ($r \in \mathbb{Z}_{\geqslant 0}$, $s \in \mathbb{Z}_{\geqslant 1}$) case, we have

$$\mathcal{T}_{\lambda\subset\mu}(u) = \det_{1\leqslant i,j\leqslant \mu_1} (\mathcal{T}^{\mu_i'-\lambda_j'-i+j}(u-\mu_1+\mu_1'-\mu_i'-\lambda_j'+i+j-1)) \tag{3.39}$$

$$= \det_{1\leqslant i,j\leqslant \mu_1'} (\mathcal{T}_{\mu_j-\lambda_i+i-j}(u-\mu_1+\mu_1'+\mu_j+\lambda_i-i-j+1)). \tag{3.40}$$

For the $D(r|s)$ ($r \in \mathbb{Z}_{\geqslant 2}$, $s \in \mathbb{Z}_{\geqslant 1}$) case, we have

$$\mathcal{T}_m(u) = \det_{1\leqslant i,j\leqslant m} (\mathcal{T}^{1-i+j}(u-m+i+j-1)). \tag{3.41}$$

Note that the function $\mathcal{T}^1(u) = \mathcal{T}_1(u)$ coincides with the eigenvalue formula of a $B(r|s)$ or $D(r|s)$ vertex model by the algebraic Bethe ansatz [16] after some redefinitions.

We remark that if $\Phi(-u) = \pm\Phi(u)$, $\boxed{a}_u$ is transformed to $\boxed{\overline{a}}_u$‖ under the following transformation.

The $B(r|s)$ ($r \in \mathbb{Z}_{\geqslant 0}$, $s \in \mathbb{Z}_{\geqslant 1}$) case:

$$\begin{aligned} &u \to -(u+2r-2s-1) \\ &u_j^{(a)} \to -u_j^{(a)} \\ &w_j \to -w_j. \end{aligned} \tag{3.42}$$

The $D(r|s)$ ($r \in \mathbb{Z}_{\geqslant 2}$, $s \in \mathbb{Z}_{\geqslant 1}$) case:

$$\begin{aligned} &u \to -(u+2r-2s-2) \\ &u_j^{(a)} \to -u_j^{(a)} \\ &w_j \to -w_j. \end{aligned} \tag{3.43}$$

$\mathcal{T}_m(u)$ and $\mathcal{T}^a(u)$ are invariant under the transformations (3.42) or (3.43). This invariance may be viewed as a kind of crossing symmetry.

Now we shall present examples of (3.38) for the $B(2|1)$, $J_- = \{1, \overline{1}\}$, $J_+ = \{2, 3, 0, \overline{3}, \overline{2}\}$ case:

$$\mathcal{T}^1(u) = -\boxed{1} + \boxed{2} + \boxed{3} + \boxed{0} + \boxed{\overline{3}} + \boxed{\overline{2}} - \boxed{\overline{1}} = -\phi(-2+u)\phi(1+u)\frac{Q_1(1+u)}{Q_1(-1+u)}$$
$$+\phi(u)\phi(1+u)\frac{Q_1(1+u)Q_2(-2+u)}{Q_1(-1+u)Q_2(u)} + \phi(u)\phi(1+u)\frac{Q_1(u)Q_2(3+u)}{Q_1(2+u)Q_2(1+u)}$$

† In contrast to the $B(r|s)$ case, the admissible condition for the $D(r|s)$ case has non-local nature. This property makes it difficult to extend the admissible condition for $D(r|s)$ to more general skew-Young (super) diagrams.

‡ We assume that they are finite-dimensional modules of quantum affine superalgebras (or super Yangians) [45, 46]. Thus $\mathcal{T}_{\lambda\subset\mu}(u)$ is expected to be a kind of a (super) character of such algebras. At present, we cannot, in general, justify these speculations mathematically, since we lack a systematic representation theory of such algebras. We hope that mathematically satisfactory account of our formulae appear after the development of a representation theory in the future.

§ $\mathcal{T}_m^a(u) := \mathcal{T}_{(m^a)}(u)$; $\mathcal{T}_m(u) := \mathcal{T}_m^1(u)$; $\mathcal{T}^a(u) := \mathcal{T}_1^a(u)$; $\mathcal{T}_m^0(u) = \mathcal{T}_0^a(u) = 1$ for $m, a \in \mathbb{Z}_{\geqslant 0}$; $\mathcal{T}_m^a(u) = 0$ if $m \in \mathbb{Z}_{<0}$ or $a \in \mathbb{Z}_{<0}$. See also appendix B.

‖ Here we interpret $\overline{0}$ as 0.

$$+\phi(u)\phi(1+u)\frac{Q_2(2+u)Q_3(-1+u)}{Q_2(u)Q_3(1+u)}+\phi(u)\phi(1+u)\frac{Q_2(-1+u)Q_3(2+u)}{Q_2(1+u)Q_3(u)}$$
$$+\phi(u)\phi(1+u)\frac{Q_3(-1+u)Q_3(2+u)}{Q_3(u)Q_3(1+u)}-\phi(u)\phi(3+u)\frac{Q_1(u)}{Q_1(2+u)} \qquad (3.44)$$

$$\mathcal{T}^2(u)=\begin{array}{|c|}\hline 1\\\hline 1\\\hline\end{array}-\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}-\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}-\begin{array}{|c|}\hline 1\\\hline 0\\\hline\end{array}-\begin{array}{|c|}\hline 1\\\hline \overline{3}\\\hline\end{array}-\begin{array}{|c|}\hline 1\\\hline \overline{2}\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline \overline{1}\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 0\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline \overline{3}\\\hline\end{array}$$
$$+\begin{array}{|c|}\hline 2\\\hline \overline{2}\\\hline\end{array}-\begin{array}{|c|}\hline 2\\\hline \overline{1}\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 0\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline \overline{3}\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline \overline{2}\\\hline\end{array}-\begin{array}{|c|}\hline 3\\\hline \overline{1}\\\hline\end{array}+\begin{array}{|c|}\hline 0\\\hline 0\\\hline\end{array}+\begin{array}{|c|}\hline 0\\\hline \overline{3}\\\hline\end{array}+\begin{array}{|c|}\hline 0\\\hline \overline{2}\\\hline\end{array}-\begin{array}{|c|}\hline 0\\\hline \overline{1}\\\hline\end{array}$$
$$+\begin{array}{|c|}\hline \overline{3}\\\hline \overline{2}\\\hline\end{array}-\begin{array}{|c|}\hline \overline{3}\\\hline \overline{1}\\\hline\end{array}-\begin{array}{|c|}\hline \overline{2}\\\hline \overline{1}\\\hline\end{array}+\begin{array}{|c|}\hline \overline{1}\\\hline \overline{1}\\\hline\end{array}$$
$$=\phi(-1+u)\phi(2+u)\Bigg(\phi(-3+u)\phi(u)\frac{Q_1(2+u)}{Q_1(-2+u)}$$
$$+\phi(-1+u)\phi(2+u)\frac{Q_1(-1+u)Q_1(2+u)}{Q_1(u)Q_1(1+u)}$$
$$+\phi(1+u)\phi(4+u)\frac{Q_1(-1+u)}{Q_1(3+u)}$$
$$-\phi(-1+u)\phi(u)\frac{Q_1(2+u)Q_2(-3+u)}{Q_1(-2+u)Q_2(-1+u)}$$
$$-\phi(1+u)\phi(2+u)\frac{Q_1(-1+u)Q_1(2+u)Q_2(-1+u)}{Q_1(u)Q_1(1+u)Q_2(1+u)}$$
$$-\phi(-1+u)\phi(u)\frac{Q_1(-1+u)Q_1(2+u)Q_2(2+u)}{Q_1(u)Q_1(1+u)Q_2(u)}$$
$$+\phi(u)\phi(1+u)\frac{Q_1(-1+u)Q_1(2+u)Q_2(-1+u)Q_2(2+u)}{Q_1(u)Q_1(1+u)Q_2(u)Q_2(1+u)}$$
$$-\phi(1+u)\phi(2+u)\frac{Q_1(-1+u)Q_2(4+u)}{Q_1(3+u)Q_2(2+u)}$$
$$+\phi(u)\phi(1+u)\frac{Q_1(2+u)Q_3(-2+u)}{Q_1(u)Q_3(u)}$$
$$-\phi(-1+u)\phi(u)\frac{Q_1(2+u)Q_2(1+u)Q_3(-2+u)}{Q_1(u)Q_2(-1+u)Q_3(u)}$$
$$-\phi(-1+u)\phi(u)\frac{Q_1(2+u)Q_2(-2+u)Q_3(1+u)}{Q_1(u)Q_2(u)Q_3(-1+u)}$$
$$+\phi(u)\phi(1+u)\frac{Q_1(2+u)Q_2(-2+u)Q_2(-1+u)Q_3(1+u)}{Q_1(u)Q_2(u)Q_2(1+u)Q_3(-1+u)}$$
$$-\phi(-1+u)\phi(u)\frac{Q_1(2+u)Q_3(-2+u)Q_3(1+u)}{Q_1(u)Q_3(-1+u)Q_3(u)}$$
$$+\phi(u)\phi(1+u)\frac{Q_1(2+u)Q_2(-1+u)Q_3(-2+u)Q_3(1+u)}{Q_1(u)Q_2(1+u)Q_3(-1+u)Q_3(u)}$$
$$-\phi(1+u)\phi(2+u)\frac{Q_1(-1+u)Q_2(3+u)Q_3(u)}{Q_1(1+u)Q_2(1+u)Q_3(2+u)}$$
$$+\phi(u)\phi(1+u)\frac{Q_1(-1+u)Q_2(2+u)Q_2(3+u)Q_3(u)}{Q_1(1+u)Q_2(u)Q_2(1+u)Q_3(2+u)}$$

$$
\begin{aligned}
&+\phi(u)\phi(1+u)\frac{Q_2(3+u)Q_3(-2+u)Q_3(1+u)}{Q_2(1+u)Q_3(-1+u)Q_3(2+u)}\\
&+\phi(u)\phi(1+u)\frac{Q_2(-2+u)Q_2(3+u)Q_3(u)Q_3(1+u)}{Q_2(u)Q_2(1+u)Q_3(-1+u)Q_3(2+u)}\\
&+\phi(u)\phi(1+u)\frac{Q_1(-1+u)Q_3(3+u)}{Q_1(1+u)Q_3(1+u)}\\
&-\phi(1+u)\phi(2+u)\frac{Q_1(-1+u)Q_2(u)Q_3(3+u)}{Q_1(1+u)Q_2(2+u)Q_3(1+u)}\\
&+\phi(u)\phi(1+u)\frac{Q_3(-2+u)Q_3(3+u)}{Q_3(-1+u)Q_3(2+u)}\\
&+\phi(u)\phi(1+u)\frac{Q_2(-2+u)Q_3(u)Q_3(3+u)}{Q_2(u)Q_3(-1+u)Q_3(2+u)}\\
&-\phi(1+u)\phi(2+u)\frac{Q_1(-1+u)Q_3(u)Q_3(3+u)}{Q_1(1+u)Q_3(1+u)Q_3(2+u)}\\
&+\phi(u)\phi(1+u)\frac{Q_1(-1+u)Q_2(2+u)Q_3(u)Q_3(3+u)}{Q_1(1+u)Q_2(u)Q_3(1+u)Q_3(2+u)}\Bigg).
\end{aligned}
\tag{3.45}
$$

$$
\begin{aligned}
\mathcal{T}_2(u) = &-\boxed{1\,2}-\boxed{1\,3}-\boxed{1\,0}-\boxed{1\,\bar{3}}-\boxed{1\,\bar{2}}+\boxed{1\,\bar{1}}\\
&+\boxed{2\,2}+\boxed{2\,3}+\boxed{2\,0}+\boxed{2\,\bar{3}}+\boxed{2\,\bar{2}}-\boxed{2\,\bar{1}}\\
&+\boxed{3\,3}+\boxed{3\,0}+\boxed{3\,\bar{3}}+\boxed{3\,\bar{2}}-\boxed{3\,\bar{1}}+\boxed{0\,\bar{3}}\\
&+\boxed{0\,\bar{2}}-\boxed{0\,\bar{1}}+\boxed{\bar{3}\,\bar{3}}+\boxed{\bar{3}\,\bar{2}}-\boxed{\bar{3}\,\bar{1}}+\boxed{\bar{2}\,\bar{2}}-\boxed{\bar{2}\,\bar{1}}\\
&=\phi(u)\phi(1+u)\Bigg(\phi(-3+u)\phi(4+u)\frac{Q_1(u)Q_1(1+u)}{Q_1(-2+u)Q_1(3+u)}\\
&-\phi(-1+u)\phi(4+u)\frac{Q_1(u)Q_1(1+u)Q_2(-3+u)}{Q_1(-2+u)Q_1(3+u)Q_2(-1+u)}\\
&+\phi(-1+u)\phi(2+u)\frac{Q_1(2+u)Q_2(-3+u)}{Q_1(-2+u)Q_2(1+u)}\\
&-\phi(-3+u)\phi(2+u)\frac{Q_1(2+u)Q_2(-1+u)}{Q_1(-2+u)Q_2(1+u)}\\
&-\phi(-1+u)\phi(4+u)\frac{Q_1(-1+u)Q_2(2+u)}{Q_1(3+u)Q_2(u)}\\
&+\phi(-1+u)\phi(2+u)\frac{Q_1(-1+u)Q_2(4+u)}{Q_1(3+u)Q_2(u)}\\
&-\phi(-3+u)\phi(2+u)\frac{Q_1(u)Q_1(1+u)Q_2(4+u)}{Q_1(-2+u)Q_1(3+u)Q_2(2+u)}\\
&+\phi(-1+u)\phi(2+u)\frac{Q_1(u)Q_1(1+u)Q_2(-3+u)Q_2(4+u)}{Q_1(-2+u)Q_1(3+u)Q_2(-1+u)Q_2(2+u)}\\
&-\phi(-1+u)\phi(4+u)\frac{Q_1(1+u)Q_2(1+u)Q_3(-2+u)}{Q_1(3+u)Q_2(-1+u)Q_3(u)}\\
&+\phi(-1+u)\phi(2+u)\frac{Q_1(1+u)Q_2(1+u)Q_2(4+u)Q_3(-2+u)}{Q_1(3+u)Q_2(-1+u)Q_2(2+u)Q_3(u)}\\
&-\phi(-1+u)\phi(4+u)\frac{Q_1(1+u)Q_2(-2+u)Q_3(1+u)}{Q_1(3+u)Q_2(u)Q_3(-1+u)}
\end{aligned}
$$

$$
\begin{aligned}
&+\phi(-1+u)\phi(2+u)\frac{Q_1(1+u)Q_2(-2+u)Q_2(4+u)Q_3(1+u)}{Q_1(3+u)Q_2(u)Q_2(2+u)Q_3(-1+u)}\\
&-\phi(-1+u)\phi(4+u)\frac{Q_1(1+u)Q_3(-2+u)Q_3(1+u)}{Q_1(3+u)Q_3(-1+u)Q_3(u)}\\
&+\phi(-1+u)\phi(2+u)\frac{Q_1(1+u)Q_2(4+u)Q_3(-2+u)Q_3(1+u)}{Q_1(3+u)Q_2(2+u)Q_3(-1+u)Q_3(u)}\\
&+\phi(-1+u)\phi(2+u)\frac{Q_2(3+u)Q_3(-2+u)}{Q_2(-1+u)Q_3(2+u)}\\
&-\phi(-3+u)\phi(2+u)\frac{Q_1(u)Q_2(3+u)Q_3(u)}{Q_1(-2+u)Q_2(1+u)Q_3(2+u)}\\
&+\phi(-1+u)\phi(2+u)\frac{Q_1(u)Q_2(-3+u)Q_2(3+u)Q_3(u)}{Q_1(-2+u)Q_2(-1+u)Q_2(1+u)Q_3(2+u)}\\
&+\phi(-1+u)\phi(2+u)\frac{Q_2(-2+u)Q_3(3+u)}{Q_2(2+u)Q_3(-1+u)}\\
&+\phi(-1+u)\phi(2+u)\frac{Q_2(u)Q_3(-2+u)Q_3(3+u)}{Q_2(2+u)Q_3(-1+u)Q_3(u)}\\
&-\phi(-3+u)\phi(2+u)\frac{Q_1(u)Q_2(u)Q_3(3+u)}{Q_1(-2+u)Q_2(2+u)Q_3(1+u)}\\
&+\phi(-1+u)\phi(2+u)\frac{Q_1(u)Q_2(-3+u)Q_2(u)Q_3(3+u)}{Q_1(-2+u)Q_2(-1+u)Q_2(2+u)Q_3(1+u)}\\
&+\phi(-1+u)\phi(2+u)\frac{Q_2(u)Q_2(1+u)Q_3(-2+u)Q_3(3+u)}{Q_2(-1+u)Q_2(2+u)Q_3(u)Q_3(1+u)}\\
&+\phi(-1+u)\phi(2+u)\frac{Q_2(1+u)Q_3(-2+u)Q_3(3+u)}{Q_2(-1+u)Q_3(1+u)Q_3(2+u)}\\
&-\phi(-3+u)\phi(2+u)\frac{Q_1(u)Q_3(u)Q_3(3+u)}{Q_1(-2+u)Q_3(1+u)Q_3(2+u)}\\
&+\phi(-1+u)\phi(2+u)\frac{Q_1(u)Q_2(-3+u)Q_3(u)Q_3(3+u)}{Q_1(-2+u)Q_2(-1+u)Q_3(1+u)Q_3(2+u)}\Bigg). \qquad (3.46)
\end{aligned}
$$

Thanks to theorem 3.1 (see later) and the relation (3.39), these DVFs are free of poles under the following BAE:

$$
\begin{aligned}
\frac{\phi(u_k^{(1)}-1)}{\phi(u_k^{(1)}+1)} &= \frac{Q_2(u_k^{(1)}-1)}{Q_2(u_k^{(1)}+1)} \qquad \text{for} \quad 1 \leqslant k \leqslant N_1\\
-1 &= \frac{Q_1(u_k^{(2)}-1)Q_2(u_k^{(2)}+2)Q_3(u_k^{(2)}-1)}{Q_1(u_k^{(2)}+1)Q_2(u_k^{(2)}-2)Q_3(u_k^{(2)}+1)} \qquad \text{for} \quad 1 \leqslant k \leqslant N_2\\
-1 &= \frac{Q_2(u_k^{(3)}-1)Q_3(u_k^{(3)}+1)}{Q_2(u_k^{(3)}+1)Q_3(u_k^{(3)}-1)} \qquad \text{for} \quad 1 \leqslant k \leqslant N_3. \qquad (3.47)
\end{aligned}
$$

Note that DVFs have so-called *Bethe-strap* structures [26, 28], which bear resemblance to weight space diagrams. We have observed for many examples that $\mathcal{T}_{\lambda\subset\mu}(u)$ coincides with the Bethe-strap of the minimal connected component (cf [47]) which include the top term [26, 27] as the examples in figures 4–6†. The top term of $\mathcal{T}_{\lambda\subset\mu}(u)$ carries a $B(r|s)$ or $D(r|s)$ weight.

† Recently we have found curious terms (pseudo-top terms) in many Bethe straps (cf [22]). However, we have confirmed for several examples the fact that the pseudo-top terms do not influence the connectivity of the Bethe straps (cf [26, 27, 47]) in the whole.

$$-\boxed{1}\xrightarrow{(1,1)}\boxed{2}\xrightarrow{(2,0)}\boxed{3}\xrightarrow{(3,-1)}\boxed{0}\xrightarrow{(3,0)}\boxed{\overline{3}}\xrightarrow{(2,-1)}\boxed{\overline{2}}\xrightarrow{(1,-2)}\boxed{\overline{1}}$$

Figure 4. The Bethe-strap structure of $\mathcal{T}^1(u)$ (3.44) for $B(2|1) = \mathrm{osp}(5|2)$. The pair (a, b) denotes the common pole $u_k^{(a)} + b$ of the pair of the tableaux connected by the arrow. This common pole vanishes under the BAE (3.47). The leftmost tableau corresponds to the 'highest weight', which is called the *top term*. Such correspondence between a certain term in the DVF and a highest weight (to be more precise, a kind of Drinfel'd polynomial (cf [27, 44])) may be called a *top-term hypothesis* [26, 27].

For example, for the $B(r|s)$, $\lambda = \phi$, $\mu_{r+1} \leqslant s$ case, the term corresponding to the tableau

$$b(i, j) = \begin{cases} j & \text{for} \quad 1 \leqslant i \leqslant \mu'_j \qquad 1 \leqslant j \leqslant s \\ i + s & \text{for} \quad 1 \leqslant i \leqslant \mu'_j \qquad s+1 \leqslant j \leqslant \mu_1 \end{cases} \tag{3.48}$$

carries the $B(r|s)$ weight with the Kac–Dynkin label (2.5) or (2.6). The top term† [26] of the DVF (3.38) for $D(r|s)$, $\lambda = \phi$, $\mu = (1^a)$ will be

$$(-1)^a \left.\begin{array}{|c|}\hline 1 \\ \hline 1 \\ \hline \vdots \\ \hline 1 \\ \hline \end{array}\right\} a = (-1)^a \frac{Q_1(u+a)}{Q_1(u-a)} \tag{3.49}$$

which carries the $D(r|s)$ weight with the Kac–Dynkin label in (2.7). The top term‡ [26] of the DVF (3.38) for $D(r|s)$, $\lambda = \phi$, $\mu = (m^1)$ will be

$$\begin{aligned}
&(-1)^m \underbrace{\begin{array}{|c|c|c|c|}\hline 1 & 2 & \cdots & m \\ \hline \end{array}}_{m} = (-1)^m \frac{Q_m(u+1)}{Q_m(u-1)} \qquad \text{if} \quad 1 \leqslant m \leqslant s \\
&(-1)^s \underbrace{\begin{array}{|c|c|c|c|c|c|c|}\hline 1 & 2 & \cdots & s & s+1 & \cdots & s+1 \\ \hline \end{array}}_{m} \\
&\qquad = (-1)^s \frac{Q_s(u+m-s+1)Q_{s+1}(u-m+s)}{Q_s(u-m+s-1)Q_{s+1}(u+m-s)} \\
&\qquad \text{if} \quad r \geqslant 3 \quad \text{and} \quad m \geqslant s+1 \\
&\qquad = (-1)^s \frac{Q_s(u+m-s+1)Q_{s+1}(u-m+s)Q_{s+2}(u-m+s)}{Q_s(u-m+s-1)Q_{s+1}(u+m-s)Q_{s+2}(u+m-s)} \\
&\qquad \text{if} \quad r = 2 \quad \text{and} \quad m \geqslant s+1
\end{aligned} \tag{3.50}$$

which carries the $D(r|s)$ weight with the Kac–Dynkin label in (2.8).

Remark. There is a supposition (cf [27, 47]) that the auxiliary space of a transfer matrix is an irreducible one as a representation space of the Yangian (or quantum affine algebra) if the Bethe strap of the DVF is connected§ in the whole. Then a natural question arises: 'Is the Bethe strap of $\mathcal{T}_{\lambda\subset\mu}(u)$ (3.38) always connected in the whole?' The answer is no. In fact, for the $D(r|s)$ case, the Bethe strap of $\mathcal{T}^a(u)$ is not connected if $0 \leqslant r - s - 1 \leqslant a \leqslant 2(r - s - 1)$. So it is desirable to extract the minimal connected component of the Bethe strap which contains the top term (3.49) from $\mathcal{T}^a(u)$. A possible candidate is as follows:

$$\mathcal{T}^a(u) - h^a(u)\mathcal{T}^{-a+2(r-s-1)}(u) \tag{3.51}$$

† Here we omit the vacuum part.

‡ Here we omit the vacuum part.

§ Here the word 'connected' means that any terms in DVF are connected directly (or indirectly) to each other by arrows, for example, as shown in figures 4–6.

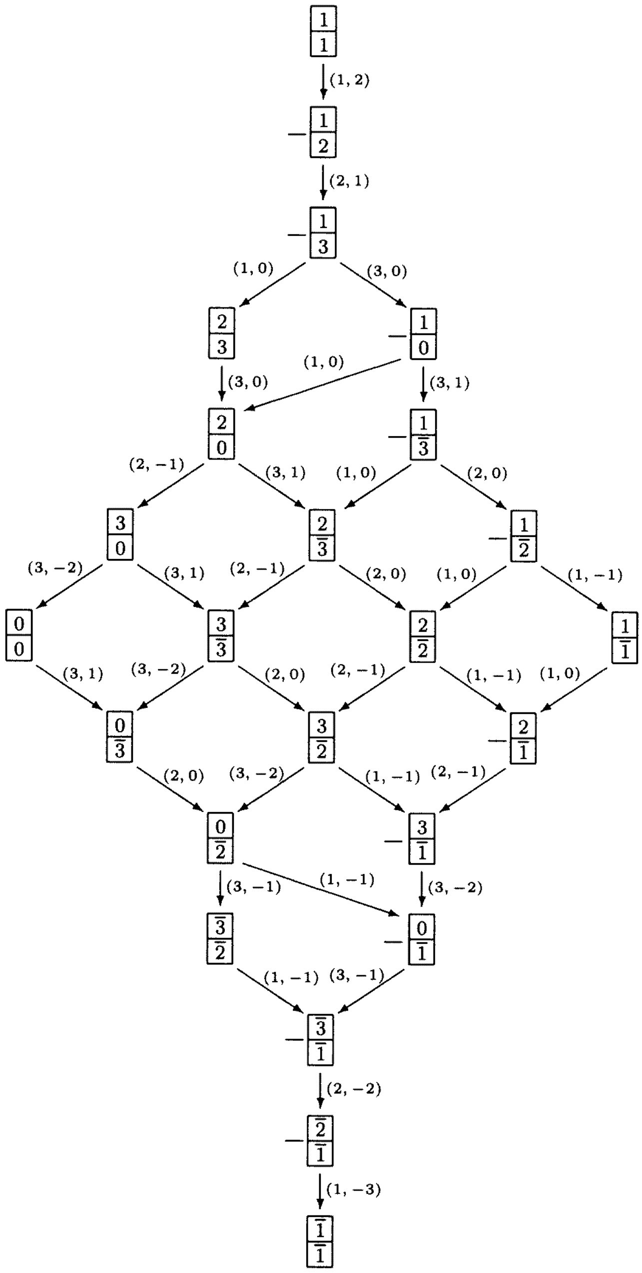

Figure 5. The Bethe-strap structure of $\mathcal{T}^2(u)$ (3.45) for $B(2|1)$. The topmost tableau corresponds to the *top term*.

where $h^a(u) = \prod_{j=1}^{a+1-r+s} \psi_1(u+a-2j+1)\psi_{\bar{1}}(u-a+2j-1)$.

For example, for the $D(3|1)$ case, $\mathcal{T}^2(u)$ consists of 31 terms and they divide into 30

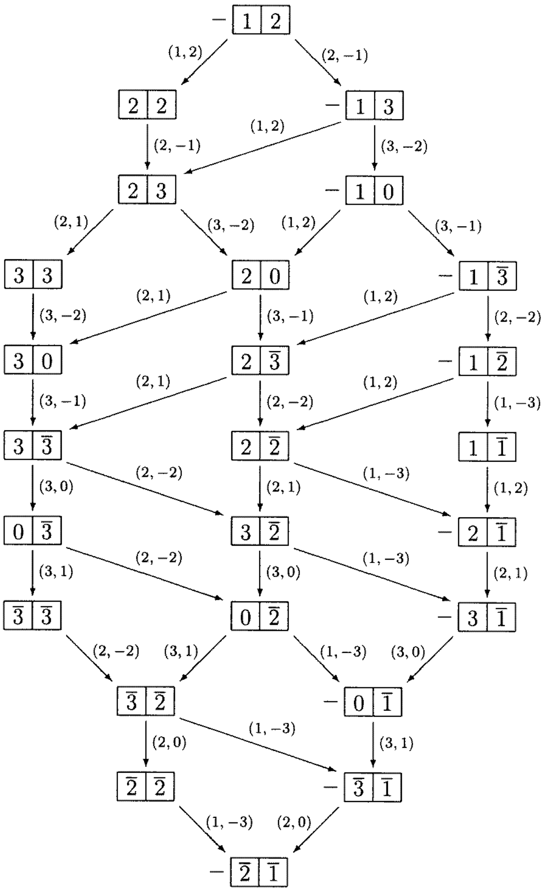

Figure 6. The Bethe-strap structure of $\mathcal{T}_2(u)$ (3.46) for $B(2|1)$. The topmost tableau corresponds to the *top term*.

terms whose Bethe strap is connected in the whole† and one isolated term $h^2(u) = \begin{array}{|c|}\hline 1 \\ \hline \bar{1} \\ \hline\end{array} = \{\phi(u-1)\phi(u+3)\}^2$. Thus the Bethe strap of $\mathcal{T}^2(u) - h^2(u)$ is connected in the whole. On the other hand, for the $D(2|2)$ case, $\mathcal{T}^2(u)$ does not have such an isolated term and the Bethe strap is connected in the whole (in this case, $\mathcal{T}^2(u)$ has 33 terms). So far, this kind of isolated term is peculiar to the $D(r|s)$ case. In fact, we have never yet observed such an isolated term in the $B(r|s)$ case. Similarly, the Bethe strap of $\mathcal{T}_m(u)$ for $D(r|s)$ seems not to be connected if $0 \leqslant s-r+1 \leqslant m \leqslant 2(s-r+1)$. A candidate for the minimal connected component of the Bethe strap which contains the top term (3.50) is

$$\mathcal{T}_m(u) - h_m(u)\mathcal{T}_{-m+2(s-r+1)}(u) \tag{3.52}$$

where $h_m(u) = \prod_{j=1}^{m+r-s-1} \psi_j(u-m+2j-1)\psi_{\bar{j}}(u+m-2j+1)$.

† In this case, $-\begin{array}{|c|}\hline 2 \\ \hline \bar{1} \\ \hline\end{array}$ is a pseudo-top term (cf [22]).

A remarkable resemblance between Bethe straps for vector representations and crystal graphs [48,49] was pointed out in [26]. Whether or not such resemblance holds true for the Lie superalgebras in general is an interesting question. There is a remarkable coincidence between currents of a deformed Virasoro algebra and DVFs [50]. Whether or not such coincidence holds true for the Lie superalgebras in general is another interesting question.

We can prove (see appendices A.1–A.3) the following theorem, which is essential in the analytic Bethe ansatz.

Theorem 3.1. *For $a \in \mathbb{Z}_{\geqslant 0}$, $\mathcal{T}^a(u)$ ((3.38) for $\lambda = \phi$, $\mu = (1^a)$) is free of poles under the condition that the BAEs (3.1)–(3.5) are valid*†.

In proving theorem 3.1, we use the following lemmas.

Lemma 3.2. *For $r \in \mathbb{Z}_{\geqslant 2}$ and $b \in \{s+1, s+2, \ldots, s+r-1\}$*

$$\begin{array}{|c|l}\hline b & _{v} \\ \hline b+1 & _{v-2} \\ \hline \end{array} \qquad \begin{array}{|c|l}\hline \overline{b+1} & _{v} \\ \hline \overline{b} & _{v-2} \\ \hline \end{array} \tag{3.53}$$

do not contain Q_b.

For the $B(0|s)$ case, we use the following lemma.

Lemma 3.3. *For $b \in \{1, 2, \ldots, s-1\}$,*

$$\begin{array}{|c|c|}\hline b & b+1 \\ \hline \end{array}_{\,u \quad u+2} \qquad \begin{array}{|c|c|}\hline \overline{b+1} & \overline{b} \\ \hline \end{array}_{\,u \quad u+2} \tag{3.54}$$

do not contain Q_b, and

$$\begin{array}{|c|c|c|}\hline s & 0 & \overline{s} \\ \hline \end{array}_{\,u \quad u+2 \quad u+4} \tag{3.55}$$

does not contain Q_s.

Then, owing to the relation (3.39), $\mathcal{T}_{\lambda\subset\mu}(u)$ for $B(r|s)$ is also free of poles under the condition that the BAEs (3.1)–(3.5) are valid. Similarly, owing to the relation (3.41), $\mathcal{T}_m(u)$ for $D(r|s)$ is also free of poles under the condition that the BAE (3.1) is valid.

4. Functional relations among DVFs

Now we introduce the functional relations among DVFs. For the $B(r|s)$ case, the following relation follows from the determinant formulae (3.39) or (3.40):

$$\mathcal{T}_m^a(u-1)\mathcal{T}_m^a(u+1) = \mathcal{T}_{m-1}^a(u)\mathcal{T}_{m+1}^a(u) + \mathcal{T}_m^{a-1}(u)\mathcal{T}_m^{a+1}(u) \tag{4.1}$$

where $m, a \in \mathbb{Z}_{\geqslant 1}$; $\mathcal{T}_m^0(u) = \mathcal{T}_0^a(u) = 1$. This functional relation (4.1) is a Hirota bilinear difference equation [51] and can be proved by using the Jacobi identity. There is a constraint to (4.1) which follows from the relation (cf [52,53] for the sl$(r|s)$ case): $\mathcal{T}_{\lambda\subset\mu}(u) = 0$ *if* $\lambda \subset \mu$

† We consider the case that the solutions $\{u_j^{(a)}\}$ of the BAEs (3.1)–(3.5) have 'generic' distribution: we assume that $u_i^{(a)} - u_j^{(a)} \neq (\alpha_a|\alpha_a)$ for any $i, j \in \{1, 2, \ldots, N_a\}$ and $a \in \{1, 2, \ldots, s+r\}$ $(i \neq j)$ in BAEs (3.1)-(3.5). Moreover, we assume that the colour b pole (see appendices A.1–A.3) of $\mathcal{T}^a(u)$ and the colour c pole do not coincide with each other if $b \neq c$. We will need separate consideration for the case where this assumption does not hold. We also note that a similar assumption was made in [19–22].

contains $a \times m$ rectangular subdiagram (a: the number of rows, m: the number of columns) with $m \in \mathbb{Z}_{\geqslant 2s+2}$ and $a \in \mathbb{Z}_{\geqslant 2r+1}$. In particular, we have

$$\mathcal{T}^a_m(u) = 0 \qquad \text{if} \quad m \in \mathbb{Z}_{\geqslant 2s+2} \quad \text{and} \quad a \in \mathbb{Z}_{\geqslant 2r+1}. \tag{4.2}$$

We also note that the determinant formula (3.41) for $D(r|s)$ reduces to the following functional relation:

$$\mathcal{T}^1(u-1)\mathcal{T}^1(u+1) = \mathcal{T}_2(u) + \mathcal{T}^2(u) \tag{4.3}$$

if we set $m = 2$.

In this section, we consider only the $B(0|s)$ ($s \in \mathbb{Z}_{\geqslant 1}$) case from now on. Now we redefine the function $\mathcal{T}_{\lambda\subset\mu}(u)$ as follows:

$$\mathcal{T}_{\lambda\subset\mu}(u) := \mathcal{T}_{\lambda\subset\mu}(u) \Big/ \left\{ \prod_{j=1}^{\mu'_1} \mathcal{F}_{\mu_j-\lambda_j}(u - \mu_1 + \mu'_1 + \mu_j + \lambda_j - 2j + 1) \right\} \tag{4.4}$$

where

$$\mathcal{F}_m(u) = \prod_{j=1}^{m-1} \phi(u - m + 2j + 1)\phi(u - 2s - m + 2j - 2) \qquad \text{for} \quad m \in \mathbb{Z}_{\geqslant 2} \tag{4.5}$$

and

$$\mathcal{F}_1(u) = 1 \qquad \mathcal{F}_0(u) = \{\phi(u+1)\phi(u-2s-2)\}^{-1}. \tag{4.6}$$

In particular, we have

$$\mathcal{T}_m(u) := \mathcal{T}_m(u)/\mathcal{F}_m(u) \tag{4.7}$$

$$\begin{aligned} \mathcal{T}^a_0(u) &= \prod_{j=1}^{a} \mathcal{T}_0(u + a - 2j + 1) \\ &= \prod_{j=1}^{a} \phi(u + a - 2j + 2)\phi(u + a - 2j - 2s - 1). \end{aligned} \tag{4.8}$$

There is a remarkable duality for $\mathcal{T}_m(u)$.

Theorem 4.1. *For any $m \in \{0, 1, \ldots, 2s+1\}$, we have*

$$\mathcal{T}_m(u) = \mathcal{T}_{2s-m+1}(u). \tag{4.9}$$

Outline of the proof. At first, we consider the case that the vacuum parts are formally trivial. In proving the relation (4.9), we use the following relations, which can be verified by direct computation:

$$\underset{u}{\boxed{\overline{a}}} \times \underset{u-2s-1}{\boxed{1}}\underset{u-2s+1}{\boxed{2}}\boxed{\cdots\cdots}\underset{u+2a-2s-3}{\boxed{a}} = \underset{u-2s+1}{\boxed{1}}\underset{u-2s+3}{\boxed{2}}\boxed{\cdots\cdots}\underset{u+2a-2s-3}{\boxed{a-1}} \tag{4.10}$$

$$\underset{u}{\boxed{a}} \times \underset{u-2a+2s+3}{\boxed{\overline{a}}}\boxed{\cdots\cdots}\underset{u+2s-1}{\boxed{\overline{2}}}\underset{u+2s+1}{\boxed{\overline{1}}} = \underset{u-2a+2s+3}{\boxed{\overline{a-1}}}\boxed{\cdots\cdots}\underset{u+2s-3}{\boxed{\overline{2}}}\underset{u+2s-1}{\boxed{\overline{1}}} \tag{4.11}$$

and

$$\underset{u-2s}{\boxed{1}}\underset{u-2s+2}{\boxed{2}}\boxed{\cdots\cdots}\underset{u-2}{\boxed{s}}\underset{u}{\boxed{0}}\underset{u+2}{\boxed{\overline{s}}}\boxed{\cdots\cdots}\underset{u-2s-2}{\boxed{\overline{2}}}\underset{u+2s}{\boxed{\overline{1}}} = 1 \tag{4.12}$$

where $a \in \{1, 2, \ldots, s+1\}$†; the spectral parameter increases (cf (3.38)) as we go from the left to the right on each tableau. We will show that any term in $\mathcal{T}_m(u)$ coincides with a term in $\mathcal{T}_{2s-m+1}(u)$. We will consider the signs originated from the grading parameter (3.12) separately. Any term in $\mathcal{T}_m(u)$ can be expressed by a tableau $b \in B((m^1))$ such that $b(1,k) = i_k$ for $1 \leqslant k \leqslant \alpha$ ($1 \preceq i_1 \prec \cdots \prec i_\alpha \preceq s$); $b(1,k) = \overline{j_{m-k+1}}$ for $\alpha+1 \leqslant k \leqslant m$ ($0 \preceq \overline{j_{m-\alpha}} \prec \cdots \prec \overline{j_1} \preceq \overline{1}$); $\alpha \in \mathbb{Z}$. The term corresponding to this tableau is given as follows‡:

$$
\underset{u-m+1}{\boxed{i_1}}\boxed{\cdots\cdots}\underset{u-m+2\alpha-1}{\boxed{i_\alpha}}\underset{u-m+2\alpha+1}{\boxed{\overline{j_{m-\alpha}}}}\boxed{\cdots\cdots}\underset{u+m-1}{\boxed{\overline{j_1}}} \tag{4.13}
$$

$$
\begin{aligned}
&= \underset{u-m+1}{\boxed{i_1}}\boxed{\cdots\cdots}\underset{u-m+2\alpha-1}{\boxed{i_\alpha}}\underset{u-m+2\alpha+1}{\boxed{\overline{j_{m-\alpha}}}}\boxed{\cdots\cdots}\underset{u+m-1}{\boxed{\overline{j_1}}} \\
&\quad\times \underset{u-m+2\alpha-2s}{\boxed{1}}\boxed{2}\boxed{\cdots\cdots}\boxed{s}\boxed{0}\boxed{\overline{s}}\boxed{\cdots\cdots}\boxed{\overline{2}}\underset{u-m+2\alpha+2s}{\boxed{\overline{1}}}
\end{aligned} \tag{4.14}
$$

$$
\begin{aligned}
&= \underset{u-m+1}{\boxed{i_1}}\boxed{\cdots\cdots}\underset{u-m+2\alpha-3}{\boxed{i_{\alpha-1}}} \times \underset{u-m+2\alpha+3}{\boxed{\overline{j_{m-\alpha-1}}}}\boxed{\cdots\cdots}\underset{u+m-1}{\boxed{\overline{j_1}}} \\
&\quad\times \underset{u-m+2\alpha+1}{\boxed{\overline{j_{m-\alpha}}}} \times \underset{u-m+2\alpha-2s}{\boxed{1}}\boxed{2}\boxed{\cdots\cdots}\underset{u-m+2\alpha-2s+2j_{m-\alpha}-2}{\boxed{j_{m-\alpha}}} \\
&\quad\times \underset{u-m+2\alpha-2s+2j_{m-\alpha}}{\boxed{j_{m-\alpha}+1}}\boxed{\cdots\cdots}\boxed{s}\boxed{0}\boxed{\overline{s}}\boxed{\cdots\cdots}\underset{u-m+2\alpha+2s-2i_\alpha}{\boxed{\overline{i_\alpha+1}}} \\
&\quad\times \underset{u-m+2\alpha-1}{\boxed{i_\alpha}} \times \underset{u-m+2\alpha-2i_\alpha+2s+2}{\boxed{\overline{i_\alpha}}}\boxed{\cdots\cdots}\boxed{\overline{2}}\underset{u-m+2\alpha+2s}{\boxed{\overline{1}}}
\end{aligned} \tag{4.15}
$$

$$
\begin{aligned}
&= \underset{u-m+1}{\boxed{i_1}}\boxed{\cdots\cdots}\underset{u-m+2\alpha-3}{\boxed{i_{\alpha-1}}} \times \underset{u-m+2\alpha+3}{\boxed{\overline{j_{m-\alpha-1}}}}\boxed{\cdots\cdots}\underset{u+m-1}{\boxed{\overline{j_1}}} \\
&\quad\times \underset{u-m+2\alpha-2s+2}{\boxed{1}}\boxed{2}\boxed{\cdots\cdots}\underset{u-m+2\alpha-2s+2j_{m-\alpha}-2}{\boxed{j_{m-\alpha}-1}} \\
&\quad\times \underset{u-m+2\alpha-2s+2j_{m-\alpha}}{\boxed{j_{m-\alpha}+1}}\boxed{\cdots\cdots}\boxed{s}\boxed{0}\boxed{\overline{s}}\boxed{\cdots\cdots}\underset{u-m+2\alpha+2s-2i_\alpha}{\boxed{\overline{i_\alpha+1}}}
\end{aligned}
$$

† Here we assume $\boxed{s+1} = \boxed{\overline{s+1}} = \boxed{0}$.

‡ (4.13) is 1 if $m = 0$.

$$\times \boxed{\overline{i_\alpha - 1}}_{u-m+2\alpha-2i_\alpha+2s+2} \boxed{\cdots\cdots} \boxed{\overline{2}} \boxed{\overline{1}}_{u-m+2\alpha+2s-2} \tag{4.16}$$

$$= \cdots = \boxed{J_1}_{u+m-2s} \boxed{\cdots\cdots} \boxed{J_{s+1-m+\alpha}}_{u-m+2\alpha} \boxed{\overline{I_{s-\alpha}}}_{u-m+2\alpha+2} \boxed{\cdots\cdots} \boxed{\overline{I_1}}_{u-m+2s} \tag{4.17}$$

where $\{J_k\} = \{1, 2, \ldots, s, 0\} \setminus \{j_1, j_2, \ldots, j_{m-\alpha}\}$ $(1 \preceq J_1 \prec \cdots \prec J_{s+1+\alpha-m} \preceq 0)$; $\{\overline{I_k}\} = \{\overline{s}, \overline{s-1}, \ldots, \overline{1}\} \setminus \{\overline{i_\alpha}, \overline{i_{\alpha-1}}, \ldots, \overline{i_1}\}$ $(\overline{s} \preceq \overline{I_{s-\alpha}} \prec \cdots \prec \overline{I_1} \preceq \overline{1})$. (4.14) follows from (4.12); (4.16) follows from (4.10) and (4.11). After repetition of procedures similar to (4.15)–(4.16), we obtain (4.17). Apparently, (4.17) is a term in $\mathcal{T}_{2s-m+1}(u)$. Conversely, one can also show that any term in $\mathcal{T}_{2s-m+1}(u)$ coincides with a term in $\mathcal{T}_m(u)$.

Noting the relation

$$\begin{aligned} \sum_{k=1}^{s+1-m+\alpha} p(J_k) + \sum_{k=1}^{s-\alpha} p(\overline{I_k}) &= \left(\sum_{k=1}^{s} p(k) + p(0) - \sum_{k=1}^{m-\alpha} p(j_k) \right) + \left(\sum_{k=1}^{s} p(\overline{k}) - \sum_{k=1}^{\alpha} p(\overline{i_k}) \right) \\ &= 2s - \sum_{k=1}^{m-\alpha} p(j_k) - \sum_{k=1}^{\alpha} p(\overline{i_k}) \equiv \sum_{k=1}^{m-\alpha} p(j_k) + \sum_{k=1}^{\alpha} p(\overline{i_k}) \bmod 2 \end{aligned} \tag{4.18}$$

we find that the overall sign for (4.13) coincides with that for (4.17).

Finally, we comment on the vacuum parts. From now on, we assume that the vacuum parts are not trivial. Equivalence between the dress parts of $\mathcal{T}_m(u)$ and $\mathcal{T}_{2s-m+1}(u)$ has already been shown, so we have only to check that the vacuum part of

$$\boxed{i_1}_{u-m+1} \boxed{\cdots\cdots} \boxed{i_\alpha}_{u-m+2\alpha-1} \boxed{\overline{j_{m-\alpha}}}_{u-m+2\alpha+1} \boxed{\cdots\cdots} \boxed{\overline{j_1}}_{u+m-1} / \mathcal{F}_m(u) \tag{4.19}$$

is equivalent to that of

$$\boxed{J_1}_{u+m-2s} \boxed{\cdots\cdots} \boxed{J_{s+1-m+\alpha}}_{u-m+2\alpha} \boxed{\overline{I_{s-\alpha}}}_{u-m+2\alpha+2} \boxed{\cdots\cdots} \boxed{\overline{I_1}}_{u-m+2s} / \mathcal{F}_{2s-m+1}(u). \tag{4.20}$$

All we have to do is to check this by direct computation for the following four cases: (i) $i_1 = 1$ and $\overline{j_1} = \overline{1}$ $(1 \prec J_1$ and $\overline{I_1} \prec \overline{1})$; (ii) $i_1 = 1$ and $\overline{j_1} \prec \overline{1}$ $(J_1 = 1$ and $\overline{I_1} \prec \overline{1})$; (iii) $1 \prec i_1$ and $\overline{j_1} = \overline{1}$ $(1 \prec J_1$ and $\overline{I_1} = \overline{1})$; (iv) $1 \prec i_1$ and $\overline{j_1} \prec \overline{1}$ $(J_1 = 1$ and $\overline{I_1} = \overline{1})$.

Owing to relation (3.40), we can generalize relation (4.9) to

$$\mathcal{T}^a_m(u) = \mathcal{T}^a_{2s-m+1}(u) \tag{4.21}$$

where $a \in \mathbb{Z}_{\geqslant 1}$. Taking note of relations (4.21) and (4.2), we shall rewrite the functional relation (4.1) in a 'canonical' form as the original T-system for the simple Lie algebra [32]. Set $T^{(a)}_m(u) = \mathcal{T}^m_a(u)$, $T^{(s)}_{2m}(u) = \mathcal{T}^m_s(u)$ and $T^{(a)}_0(u) = T^{(s)}_0(u) = T^{(0)}_m(u) = 1$ for $a \in \{1, 2, \ldots, s-1\}$ and $m \in \mathbb{Z}_{\geqslant 1}$, where the subscript (n, a) of $T^{(a)}_n(u)$ corresponds to the Kac–Dynkin label $[b_1, b_2, \ldots, b_s]$ for $b_i = n\delta_{ia}$ (cf (2.6)). Then we have

$$T^{(a)}_m(u-1)T^{(a)}_m(u+1) = T^{(a)}_{m-1}(u)T^{(a)}_{m+1}(u) + g^{(a)}_m(u)T^{(a-1)}_m(u)T^{(a+1)}_m(u) \quad \text{for} \quad a \in \{1, 2, \ldots, s-2\} \tag{4.22}$$

$$T^{(s-1)}_m(u-1)T^{(s-1)}_m(u+1) = T^{(s-1)}_{m-1}(u)T^{(s-1)}_{m+1}(u) + g^{(s-1)}_m(u)T^{(s-2)}_m(u)T^{(s)}_{2m}(u) \tag{4.23}$$

$$T^{(s)}_{2m}(u-1)T^{(s)}_{2m}(u+1) = T^{(s)}_{2m-2}(u)T^{(s)}_{2m+2}(u) + g^{(s)}_{2m}(u)T^{(s-1)}_m(u)T^{(s)}_{2m}(u) \tag{4.24}$$

where $g^{(b)}_m(u) = \{\prod_{j=1}^{m} \mathcal{T}_0(u+2j-m-1)\}^{\delta_{b,1}}$ if $s \in \mathbb{Z}_{\geqslant 2}$; $g^{(1)}_{2m}(u) = \prod_{j=1}^{m} \mathcal{T}_0(u+2j-m-1)$ if $s = 1$. Note that the function $g^{(b)}_m(u)$ obeys the following relation:

$$\begin{aligned} & g^{(b)}_m(u+1)g^{(b)}_m(u-1) = g^{(b)}_{m+1}(u)g^{(b)}_{m-1}(u) \qquad \text{if} \quad s \in \mathbb{Z}_{\geqslant 2} \\ & g^{(1)}_{2m}(u+1)g^{(1)}_{2m}(u-1) = g^{(1)}_{2m+2}(u)g^{(1)}_{2m-2}(u) \qquad \text{if} \quad s = 1. \end{aligned} \tag{4.25}$$

These functional relations (4.22)–(4.24) will be the $B(0|s)$ version of the T-system. Note that the subscript n of $T^{(s)}_n(u)$ can only take an even number (cf (2.9)). By construction, $T^{(a)}_m(u)$ can be expressed as a determinant over a matrix whose matrix elements are only the fundamental functions $T^{(1)}_1, \ldots, T^{(s-1)}_1, T^{(s)}_2$ and $g^{(1)}_1$ for $s \in \mathbb{Z}_{\geqslant 2}$; $T^{(1)}_2$ and $g^{(1)}_2$ for $s = 1$. This can be summarized as follows.

Theorem 4.2. *For* $m \in \mathbb{Z}_{\geqslant 1}$,

$$T^{(a)}_m(u) = \det_{1\leqslant i,j\leqslant m}(\mathcal{T}_{a+i-j}(u+m-i-j+1)) \qquad \textit{for} \quad a \in \{1, 2, \ldots, s-1\} \tag{4.26}$$

$$T^{(s)}_{2m}(u) = \det_{1\leqslant i,j\leqslant m}(\mathcal{T}_{s+i-j}(u+m-i-j+1)) \tag{4.27}$$

solves (4.22)–(4.24). Here $\mathcal{T}_a(u)$ *obeys the relation (4.9) and the boundary condition*

$$\mathcal{T}_a(u) = \begin{cases} 0 & \text{if} \quad a < 0 \\ g^{(1)}_1(u) & \text{if} \quad a = 0 \quad \text{and} \quad s \in \mathbb{Z}_{\geqslant 2} \\ g^{(1)}_2(u) & \text{if} \quad a = 0 \quad \text{and} \quad s = 1 \\ T^{(a)}_1(u) & \text{if} \quad a \in \{1, 2, \ldots, s-1\} \\ T^{(s)}_2(u) & \text{if} \quad a = s \end{cases} \tag{4.28}$$

where $g^{(a)}_m(u) = \{\prod_{j=1}^{m} g^{(1)}_1(u+2j-m-1)\}^{\delta_{a,1}}$ *if* $s \in \mathbb{Z}_{\geqslant 2}$; $g^{(1)}_{2m}(u) = \prod_{j=1}^{m} g^{(1)}_2(u+2j-m-1)$ *if* $s = 1$.

Remark. These functional realtions (4.22)–(4.24) resemble the ones for $A^{(2)}_{2s}$ [29]. This resemblance will originate from the resemblance between $B(0|s)^{(1)}$ and $A^{(2)}_{2s}$.

There is a remarkable relation between the number $\mathcal{N}^{(a)}_m$ of the terms in $T^{(a)}_m(u)$ and the dimensionality (2.10) of the Lie superalgebra $B(0|s)$. We conjecture that they are related each other as follows:

$$\begin{aligned} & \mathcal{N}^{(a)}_m = \sum \dim V[k_1, k_2, \ldots, k_a, 0, \ldots, 0] \qquad \text{if} \quad a \in \{1, 2, \ldots, s-1\} \\ & \mathcal{N}^{(s)}_{2m} = \sum \dim V[k_1, k_2, \ldots, k_{s-1}, 2k_s] \end{aligned} \tag{4.29}$$

where the summation is taken over non-negative integers $\{k_j\}$ such that $k_1 + k_2 + \cdots + k_a \leqslant m$ and $k_j \equiv m\delta_{ja}$ mod 2. For example, for the $B(0|2)$ case, we have (cf tables 1 and 2):

$$\begin{aligned} & \mathcal{N}^{(1)}_1 = \dim V[1, 0] \\ & \mathcal{N}^{(1)}_2 = \dim V[2, 0] + \dim V[0, 0] \\ & \mathcal{N}^{(1)}_3 = \dim V[3, 0] + \dim V[1, 0] \\ & \mathcal{N}^{(2)}_2 = \dim V[0, 2] \\ & \mathcal{N}^{(2)}_4 = \dim V[0, 4] + \dim V[2, 0] + \dim V[0, 0] \\ & \mathcal{N}^{(2)}_6 = \dim V[0, 6] + \dim V[2, 2] + \dim V[0, 2]. \end{aligned} \tag{4.30}$$

These relations seem to suggest a superization of the Kirillov–Reshetikhin formula [54], which gives the multiplicity of the occurrence of irreducible representations of the Lie superalgebra in the Yangian module.

Table 1. The number $\mathcal{N}_m^{(a)}$ of the terms in $T_m^{(a)}(u)$ for $B(0|2)$.

m	1	2	3	4
$\mathcal{N}_m^{(1)}$	5	15	35	70
$\mathcal{N}_{2m}^{(2)}$	10	50	175	490

Table 2. The dimensionality of the module $V[b_1, b_2]$ for $B(0|2)$.

$[b_1, b_2]$	dim $V[b_1, b_2]$	$[b_1, b_2]$	dim $V[b_1, b_2]$
0 0	1	0 2	10
1 0	5	0 4	35
2 0	14	0 6	84
3 0	30	2 2	81

5. Summary and discussion

In this paper, we have carried out an analytic Bethe ansatz based on the BAEs (3.1)–(3.5) with the distinguished simple root systems of the type-II Lie superalgebras $B(r|s)$ and $D(r|s)$. We proposed eigenvalue formulae of transfer matrices in DVFs related to a class of tensor-like representations, and showed their pole-freeness under the BAEs (3.1)–(3.5). The key is the top-term hypothesis and the pole-freeness under the BAE. A class of functional relations was proposed for the DVFs. In particular for the $B(0|s)$ case, remarkable duality among DVFs was found. By using this, a complete set of functional relations was written down for the DVFs labelled by rectangular Young (super) diagrams. To the author's knowledge, this paper is the first attempt to construct a *systematic* theory of an analytic Bethe ansatz related to fusion $B(r|s)$ and $D(r|s)$ vertex models.

In the present paper, we executed an analytic Bethe ansatz only for tensor-like representations. As for spinorial representations, details are under investigation. For example, in relation to the 64-dimensional typical representation of $B(2|1)$, we confirmed the fact that the Bethe-strap generated by the following top term†:

$$\frac{Q_1(u+\frac{5}{2})Q_3(u-\frac{1}{2})}{Q_1(u-\frac{5}{2})Q_3(u+\frac{1}{2})} \tag{5.1}$$

which carries $B(2|1)$ weight with the Kac–Dynkin label $(\frac{5}{2}, 0, 1)$ consists of 64 terms.

For the $D(r|s)$ case, we proposed DVFs labelled by Young (super) diagrams with one row or one column. It is tempting to extend these DVFs to general Young (super) diagrams. However, this will be a difficult task since we lack tableaux sum expressions of DVFs labelled by general Young diagrams even for the non-superalgebra D_r case [26]. One way to bypass cumbersome tableaux sum expressions is to construct a complete set of transfer matrix functional relations (a hierarchy of T-system). By solving it, we will be able to calculate DVFs.

It is an interesting problem to derive TBA equations from our T-system (4.22)–(4.24). This is accomplished by a similar procedure to the sl$(r|s)$ case [55] (see also [56]). We will report on this in the near future.

† Here we omit the vacuum part.

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Appendix A.1. Outline of the proof of theorem 3.1: the $B(r|s)$ $(r, s \in \mathbb{Z}_{\geqslant 1})$ case

For simplicity, we assume that the vacuum parts are formally trivial from now on. We prove that $\mathcal{T}^a(u)$ is free of colour b poles, that is, $\mathrm{Res}_{u=u_k^{(b)}+\cdots}\mathcal{T}^a(u) = 0$ for any $b \in \{1, 2, \ldots, s+r\}$ under the condition that the BAE (3.1) is valid. The function $\boxed{c}_u$ (3.13) with $c \in J$ has colour b poles only for $c = b, b+1, \overline{b+1}$ or $\overline{b}$ if $b \in \{1, 2, \ldots, s+r-1\}$; for $c = s+r$, 0 or $\overline{s+r}$ if $b = s+r$, so we shall trace only $\boxed{b}$, $\boxed{b+1}$, $\boxed{\overline{b+1}}$ or $\boxed{\overline{b}}$ for $b \in \{1, 2, \ldots, s+r-1\}$; $\boxed{s+r}$, $\boxed{0}$ or $\boxed{\overline{s+r}}$ for $b = s+r$. Let S_k be the partial sum of $\mathcal{T}^a(u)$, which contains k boxes among $\boxed{b}$, $\boxed{b+1}$, $\boxed{\overline{b+1}}$ or $\boxed{\overline{b}}$ for $b \in \{1, 2, \ldots, s+r-1\}$; $\boxed{s+r}$, $\boxed{0}$ or $\boxed{\overline{s+r}}$ for $b = s+r$. Evidently, S_0 does not have colour b poles.

Now we examine S_1 which is a summation of the tableaux (with sign) of the form

$$\begin{array}{|c|}\hline \xi \\ \hline \eta \\ \hline \zeta \\ \hline \end{array} \tag{A.1.1}$$

where $\boxed{\xi}$ and $\boxed{\zeta}$ are columns with total length $a-1$ and they do not involve Q_b. $\boxed{\eta}$ is $\boxed{b}$, $\boxed{b+1}$, $\boxed{\overline{b+1}}$ or $\boxed{\overline{b}}$ for $b \in \{1, 2, \ldots, s+r-1\}$; $\boxed{s+r}$, $\boxed{0}$ or $\boxed{\overline{s+r}}$ for $b = s+r$. Thanks to the relations (3.18)–(3.25), S_1 is free of colour b poles under the BAE (3.1). Hereafter we consider S_k for $k \geqslant 2$.

• The case † $b \in \{1, 2, \ldots, s-1\}$: $S_k (k \geqslant 2)$ is a summation of the tableaux (with sign) of the form

$$\sum_{n_1=0}^{k_1}\sum_{n_2=0}^{k_2} \begin{array}{|c|}\hline \xi \\ \hline E_{1n_1} \\ \hline \eta \\ \hline E_{2n_2} \\ \hline \zeta \\ \hline \end{array} = \left(\sum_{n_1=0}^{k_1} \boxed{E_{1n_1}}\right)\left(\sum_{n_2=0}^{k_2} \boxed{E_{2n_2}}\right) \times \boxed{\xi} \times \boxed{\eta} \times \boxed{\zeta} \tag{A.1.2}$$

where $\boxed{\xi}$, $\boxed{\eta}$ and $\boxed{\zeta}$ are columns with total length $a-k$, which do not contain $\boxed{b}$, $\boxed{b+1}$,

† This is void for the $B(r|1)$ case.

$\boxed{\overline{b+1}}$ and $\boxed{\overline{b}}$; $\boxed{E_{1n_1}}$ is a column† of the form

$$\begin{array}{|c|l}\hline b & {\scriptstyle v} \\ \hline \vdots & \\ \hline b & {\scriptstyle v-2n_1+2} \\ \hline b+1 & {\scriptstyle v-2n_1} \\ \hline \vdots & \\ \hline b+1 & {\scriptstyle v-2k_1+2} \\ \hline \end{array} = \frac{Q_{b-1}(v-b+1-2n_1)Q_b(v-b+2)}{Q_{b-1}(v-b+1)Q_b(v-b-2n_1+2)}$$

$$\times \frac{Q_b(v-b-2k_1)Q_{b+1}(v-b+1-2n_1)}{Q_b(v-b-2n_1)Q_{b+1}(v-b+1-2k_1)} \tag{A.1.3}$$

where $v = u + h_1$: h_1 is some shift parameter and $\boxed{E_{2n_2}}$ is a column‡ of the form

$$\begin{array}{|c|l}\hline \overline{b+1} & {\scriptstyle w} \\ \hline \vdots & \\ \hline \overline{b+1} & {\scriptstyle w-2n_2+2} \\ \hline \overline{b} & {\scriptstyle w-2n_2} \\ \hline \vdots & \\ \hline \overline{b} & {\scriptstyle w-2k_2+2} \\ \hline \end{array} = \frac{Q_{b-1}(w-2s+2r+b-2n_2)}{Q_{b-1}(w-2s+2r+b-2k_2)} \times \frac{Q_b(w-2s+2r+b-2k_2-1)}{Q_b(w-2s+2r+b-2n_2-1)}$$

$$\times \frac{Q_b(w-2s+2r+b+1)Q_{b+1}(w-2s+2r+b-2n_2)}{Q_b(w-2s+2r+b-2n_2+1)Q_{b+1}(w-2s+2r+b)} \tag{A.1.4}$$

where $w = u + h_2$: h_2 is some shift parameter; $k = k_1 + k_2$§.

For $b \in \{1, 2, \ldots, s-1\}$, $\boxed{E_{1n_1}}$ has colour b poles at $u = -h_1 + b + 2n_1 + u_p^{(b)}$ and $u = -h_1 + b + 2n_1 - 2 + u_p^{(b)}$ for $1 \leqslant n_1 \leqslant k_1 - 1$; at $u = -h_1 + b + u_p^{(b)}$ for $n_1 = 0$; at $u = -h_1 + b + 2k_1 - 2 + u_p^{(b)}$ for $n_1 = k_1$‖. The colour b residues at $u = -h_1 + b + 2n_1 + u_p^{(b)}$ in $\boxed{E_{1n_1}}$ and $\boxed{E_{1\,n_1+1}}$ cancel each other under the BAE (3.1). Thus, under the BAE (3.1), $\sum_{n_1=0}^{k_1} \boxed{E_{1n_1}}$ is free of colour b poles (see figure A.1.1).

$\boxed{E_{2n_2}}$ has colour b poles at $u = -h_2 + 2s - 2r - b + 2n_2 - 1 + u_p^{(b)}$ and $u = -h_2 + 2s - 2r - b + 2n_2 + 1 + u_p^{(b)}$ for $1 \leqslant n_2 \leqslant k_2 - 1$; at $u = -h_2 + 2s - 2r - b + 1 + u_p^{(b)}$ for $n_2 = 0$; at $u = -h_2 + 2s - 2r - b + 2k_2 - 1 + u_p^{(b)}$ for $n_2 = k_2$. The colour b residues at

† We assume that $\boxed{E_{10}} = \begin{array}{|c|l}\hline b+1 & {\scriptstyle v} \\ \hline \vdots & \\ \hline b+1 & {\scriptstyle v-2k_1+2} \\ \hline \end{array}$ and $\boxed{E_{1k_1}} = \begin{array}{|c|l}\hline b & {\scriptstyle v} \\ \hline \vdots & \\ \hline b & {\scriptstyle v-2k_1+2} \\ \hline \end{array}$.

‡ We assume that $\boxed{E_{20}} = \begin{array}{|c|l}\hline \overline{b} & {\scriptstyle w} \\ \hline \vdots & \\ \hline \overline{b} & {\scriptstyle w-2k_2+2} \\ \hline \end{array}$ and $\boxed{E_{2k_2}} = \begin{array}{|c|l}\hline \overline{b+1} & {\scriptstyle w} \\ \hline \vdots & \\ \hline \overline{b+1} & {\scriptstyle w-2k_2+2} \\ \hline \end{array}$.

§ We assume that $\boxed{E_{1n_1}} = 1$ (resp. $\boxed{E_{2n_2}} = 1$) for $k_1 = 0$ (resp. $k_2 = 0$). In this case, $\boxed{E_{in_i}}$ does not have poles.

‖ We assume that these poles at $u = -h_1 + b + 2n_1 + u_i^{(b)}$, and $u = -h_1 + b + 2n_1 - 2 + u_q^{(b)}$ do not coincide with each other for any $i, q \in \{1, 2, \ldots, N_b\}$: namely $u_i^{(b)} - u_q^{(b)} \neq 2$.

$$\boxed{E_{10}} \overset{0}{\longleftarrow} \boxed{E_{11}} \overset{2}{\longleftarrow} \cdots \overset{2n-4}{\longleftarrow} \boxed{E_{1n-1}} \overset{2n-2}{\longleftarrow} \boxed{E_{1n}} \overset{2n}{\longleftarrow} \boxed{E_{1n+1}} \overset{2n+2}{\longleftarrow} \cdots \overset{2k_1-2}{\longleftarrow} \boxed{E_{1k_1}}$$

Figure A.1.1. Partial Bethe-strap structure of E_{1n} for colour b poles ($1 \leqslant b \leqslant s-1$). The number n on the arrow denotes the common colour b pole $-h_1+b+n+u_k^{(b)}$ of the pair of tableaux connected by the arrow. This common pole vanishes under the BAE (3.1).

$u = -h_2 + 2s - 2r - b + 2n_2 + 1 + u_p^{(b)}$ in $\boxed{E_{2n_2}}$ and $\boxed{E_{2,n_2+1}}$ cancel each other out under the BAE (3.1). Thus, under the BAE (3.1), $\sum_{n_2=0}^{k_2} \boxed{E_{2,n_2}}$ is free of colour b poles and so is S_k.

- The case $b = s$: $S_k (k \geqslant 2)$ is a summation of the tableaux (with sign) of the form

$$\begin{array}{|c|}\hline D_{11} \\ \hline \eta \\ \hline D_{21} \\ \hline \end{array} - \begin{array}{|c|}\hline D_{11} \\ \hline \eta \\ \hline D_{22} \\ \hline \end{array} - \begin{array}{|c|}\hline D_{12} \\ \hline \eta \\ \hline D_{21} \\ \hline \end{array} + \begin{array}{|c|}\hline D_{12} \\ \hline \eta \\ \hline D_{22} \\ \hline \end{array} = \left(\boxed{D_{11}} - \boxed{D_{12}}\right)\left(\boxed{D_{21}} - \boxed{D_{22}}\right)\boxed{\eta} \tag{A.1.5}$$

where $\boxed{\eta}$ is a column with length $a-k$, which does not contain $\boxed{s}$, $\boxed{s+1}$, $\boxed{\overline{s+1}}$ and $\boxed{\overline{s}}$; $\boxed{D_{11}}$ is a column† of the form

$$\begin{array}{|c|l}\hline s & {\scriptstyle v} \\ \hline \vdots & \\ \hline s & {\scriptstyle v-2k_1+4} \\ \hline s+1 & {\scriptstyle v-2k_1+2} \\ \hline \end{array} = \frac{Q_{s-1}(v-s-2k_1+3)Q_s(v-s+2)Q_{s+1}(v-s-2k_1+1)}{Q_{s-1}(v-s+1)Q_s(v-s-2k_1+2)Q_{s+1}(v-s-2k_1+3)} \tag{A.1.6}$$

$\boxed{D_{12}}$ is a column of the form

$$\begin{array}{|c|l}\hline s & {\scriptstyle v} \\ \hline \vdots & \\ \hline s & {\scriptstyle v-2k_1+4} \\ \hline s & {\scriptstyle v-2k_1+2} \\ \hline \end{array} = \frac{Q_{s-1}(v-s-2k_1+1)Q_s(v-s+2)}{Q_{s-1}(v-s+1)Q_s(v-s-2k_1+2)} \tag{A.1.7}$$

where $v = u + h_1$: h_1 is some shift parameter; $\boxed{D_{21}}$ is a column‡ of the form

$$\begin{array}{|c|l}\hline \overline{s+1} & {\scriptstyle w} \\ \hline \overline{s} & {\scriptstyle w-2} \\ \hline \vdots & \\ \hline \overline{s} & {\scriptstyle w-2k_2+2} \\ \hline \end{array} = \frac{Q_{s-1}(w-s+2r-2)}{Q_{s-1}(w-s+2r-2k_2)} \times \frac{Q_s(w-s+2r-2k_2-1)Q_{s+1}(w-s+2r)}{Q_s(w-s+2r-1)Q_{s+1}(w-s+2r-2)} \tag{A.1.8}$$

† We assume that $\boxed{D_{11}} = \boxed{s+1}_v$ if $k_1 = 1$.

‡ We assume that $\boxed{D_{21}} = \boxed{\overline{s+1}}_w$ if $k_2 = 1$.

$\boxed{D_{22}}$ is a column of the form

$$\begin{array}{|c|l}\hline \overline{s} & _{w} \\ \hline \overline{s} & _{w-2} \\ \hline \vdots & \\ \hline \overline{s} & _{w-2k_2+2} \\ \hline \end{array} = \frac{Q_{s-1}(w-s+2r)Q_s(w-s+2r-2k_2-1)}{Q_{s-1}(w-s+2r-2k_2)Q_s(w-s+2r-1)} \tag{A.1.9}$$

where $w = u + h_2$: h_2 is some shift parameter; $k = k_1 + k_2$ †. Obviously, the colour $b = s$ residues at $v = s + 2k_1 - 2 + u_j^{(s)}$ in (A.1.6) and (A.1.7) cancel each other out under the BAE (3.1). And the colour $b = s$ residues at $w = s - 2r + 1 + u_j^{(s)}$ in (A.1.8) and (A.1.9) cancel each other out under the BAE (3.1). Thus S_k does not have colour s poles under the BAE (3.1).

• The case‡ $b \in \{s+1, s+2, \ldots, s+r-1\}$: Owing to the admissibility conditions, we have only to consider S_k for $k = 2, 3, 4$.

S_2 is a summation of the tableaux (with sign) of the form

$$\begin{array}{|c|}\hline \xi \\ \hline b \\ \hline b+1 \\ \hline \zeta' \\ \hline \end{array} + \begin{array}{|c|}\hline \xi' \\ \hline \overline{b+1} \\ \hline \overline{b} \\ \hline \zeta \\ \hline \end{array} \tag{A.1.10}$$

and

$$\begin{array}{|c|}\hline \xi \\ \hline b \\ \hline \eta \\ \hline \overline{b} \\ \hline \zeta \\ \hline \end{array} + \begin{array}{|c|}\hline \xi \\ \hline b \\ \hline \eta \\ \hline \overline{b+1} \\ \hline \zeta \\ \hline \end{array} + \begin{array}{|c|}\hline \xi \\ \hline b+1 \\ \hline \eta \\ \hline \overline{b} \\ \hline \zeta \\ \hline \end{array} + \begin{array}{|c|}\hline \xi \\ \hline b+1 \\ \hline \eta \\ \hline \overline{b+1} \\ \hline \zeta \\ \hline \end{array} = \boxed{\xi}\left(\boxed{b} + \boxed{b+1}\right)\boxed{\eta}\left(\boxed{\overline{b}} + \boxed{\overline{b+1}}\right)\boxed{\zeta} \tag{A.1.11}$$

where $\{\boxed{\xi}, \boxed{\eta}, \boxed{\zeta}\}$, $\{\boxed{\xi}, \boxed{\zeta'}\}$ and $\{\boxed{\xi'}, \boxed{\zeta}\}$ are columns with total length $a - 2$, which do not contain $\boxed{b}$, $\boxed{b+1}$, $\boxed{\overline{b+1}}$ and $\boxed{\overline{b}}$. Thus, owing to lemma 3.2, the relations (3.20) and (3.23), S_2 does not have colour b poles under the BAE (3.1).

S_3 is a summation of the tableaux (with sign) of the form

$$\begin{array}{|c|}\hline \xi \\ \hline b \\ \hline \eta \\ \hline \overline{b+1} \\ \hline \overline{b} \\ \hline \zeta \\ \hline \end{array} + \begin{array}{|c|}\hline \xi \\ \hline b+1 \\ \hline \eta \\ \hline \overline{b+1} \\ \hline \overline{b} \\ \hline \zeta \\ \hline \end{array} + \begin{array}{|c|}\hline \xi \\ \hline b \\ \hline b+1 \\ \hline \eta' \\ \hline \overline{b} \\ \hline \zeta \\ \hline \end{array} + \begin{array}{|c|}\hline \xi \\ \hline b \\ \hline b+1 \\ \hline \eta' \\ \hline \overline{b+1} \\ \hline \zeta \\ \hline \end{array} \tag{A.1.12}$$

where $\{\boxed{\xi}, \boxed{\eta}, \boxed{\zeta}\}$ and $\{\boxed{\xi}, \boxed{\eta'}, \boxed{\zeta}\}$ are columns with total length $a - 3$, which do not contain $\boxed{b}$, $\boxed{b+1}$, $\boxed{\overline{b+1}}$ and $\boxed{\overline{b}}$. Thus, owing to relations (3.20), (3.23) and lemma 3.2, S_3 does not have colour b poles under the BAE (3.1).

† Here we discussed the case for $k_1 \geqslant 1$ and $k_2 \geqslant 1$; the case for $k_1 = 0$ or $k_2 = 0$ can be treated similarly.
‡ This is void for the $B(1|s)$ case.

S_4 is a summation of the tableaux (with sign) of the form

$$\begin{array}{|c|}\hline \xi \\ \hline b \\ \hline b+1 \\ \hline \eta \\ \hline \overline{b+1} \\ \hline \overline{b} \\ \hline \zeta \\ \hline \end{array} \tag{A.1.13}$$

where $\boxed{\xi}$, $\boxed{\eta}$ and $\boxed{\zeta}$ are columns with total length $a-4$, which do not contain $\boxed{b}$, $\boxed{b+1}$, $\boxed{\overline{b+1}}$ and $\boxed{\overline{b}}$. Thus, owing to lemma 3.2, S_4 does not have colour b poles under the BAE (3.1).

• The case $b = s + r$: S_k $(k \geqslant 2)$ is a summation of the tableaux (with sign) of the form

$$\begin{array}{|c|l}\hline \xi & \\ \hline 0 & {\scriptstyle v} \\ \hline 0 & {\scriptstyle v-2} \\ \hline \vdots & \\ \hline 0 & {\scriptstyle v-2k+4} \\ \hline 0 & {\scriptstyle v-2k+2} \\ \hline \zeta & \\ \hline \end{array} + \begin{array}{|c|l}\hline \xi & \\ \hline s+r & {\scriptstyle v} \\ \hline 0 & {\scriptstyle v-2} \\ \hline \vdots & \\ \hline 0 & {\scriptstyle v-2k+4} \\ \hline 0 & {\scriptstyle v-2k+2} \\ \hline \zeta & \\ \hline \end{array} + \begin{array}{|c|l}\hline \xi & \\ \hline 0 & {\scriptstyle v} \\ \hline 0 & {\scriptstyle v-2} \\ \hline \vdots & \\ \hline 0 & {\scriptstyle v-2k+4} \\ \hline \overline{s+r} & {\scriptstyle v-2k+2} \\ \hline \zeta & \\ \hline \end{array} + \begin{array}{|c|l}\hline \xi & \\ \hline s+r & {\scriptstyle v} \\ \hline 0 & {\scriptstyle v-2} \\ \hline \vdots & \\ \hline 0 & {\scriptstyle v-2k+4} \\ \hline \overline{s+r} & {\scriptstyle v-2k+2} \\ \hline \zeta & \\ \hline \end{array}$$

$$= A(v)B(v) \times \boxed{\xi} \times \boxed{\eta} \tag{A.1.14}$$

where $v = u + h_3$: h_3 is some shift parameter; $\boxed{\xi}$ and $\boxed{\zeta}$ are columns with total length $a-k$, which do not contain $\boxed{s+r}$, $\boxed{0}$ and $\boxed{\overline{s+r}}$;

$$A(v) = \frac{Q_{s+r}(v-s+r+1)}{Q_{s+r}(v-s+r)} + \frac{Q_{s+r-1}(v-s+r+1)Q_{s+r}(v-s+r-1)}{Q_{s+r-1}(v-s+r-1)Q_{s+r}(v-s+r)} \tag{A.1.15}$$

$$B(v) = \frac{Q_{s+r}(v-s+r-2k)}{Q_{s+r}(v-s+r-2k+1)} + \frac{Q_{s+r-1}(v-s+r-2k)Q_{s+r}(v-s+r-2k+2)}{Q_{s+r-1}(v-s+r-2k+2)Q_{s+r}(v-s+r-2k+1)}.$$

One can check $A(v)$ and $B(v)$ are free of colour $s+r$ poles under the BAE (3.1). Thus, S_k does not have colour $s+r$ poles under the BAE (3.1).

Remark. There is another proof for theorem 3.1 by the determinant formula (3.40): for $b \in \{1, 2, \ldots, s-1\}$, we prove $\mathcal{T}_m(u)$ is free of colour b poles, and then the pole-freeness of $\mathcal{T}^a(u)$ follows from (3.40); while for $b \in \{s, s+1, \ldots, s+r\}$, we prove $\mathcal{T}^a(u)$ is free of colour b poles in the same way as the above-mentioned proof. An advantage of this alternative proof is that we do not encounter an awkward expression like (A.1.2). We note that a similar idea is also applicable for the sl$(r|s)$ case [19–21].

Appendix A.2. Outline of the proof of theorem 3.1: the $B(0|s)$ ($s \in \mathbb{Z}_{\geqslant 1}$) case

We will show that $\mathcal{T}_m(u)$ is free of colour b poles, namely, $\mathrm{Res}_{u=u_k^{(b)}+\cdots}\mathcal{T}_m(u) = 0$ for any $b \in \{1, 2, \ldots, s\}$ under the condition that the BAE (3.2)–(3.5) is valid. The function $\boxed{c}_u$

(3.13) with $c \in J$ has colour b poles only for $c = b, b+1, \overline{b+1}$ or $\overline{b}$ if $b \in \{1, 2, \ldots, s-1\}$; for $c = s$, 0 or $\overline{s}$ if $b = s$, so we shall trace only $\boxed{b}$, $\boxed{b+1}$, $\boxed{\overline{b+1}}$ or $\boxed{\overline{b}}$ for $b \in \{1, 2, \ldots, s-1\}$; $\boxed{s}$, $\boxed{0}$ or $\boxed{\overline{s}}$ for $b = s$. Let S_k be the partial sum of $\mathcal{T}_m(u)$, which contains k boxes among $\boxed{b}$, $\boxed{b+1}$, $\boxed{\overline{b+1}}$ or $\boxed{\overline{b}}$ for $b \in \{1, 2, \ldots, s-1\}$; $\boxed{s}$, $\boxed{0}$ or $\boxed{\overline{s}}$ for $b = s$. Apparently, S_0 does not have colour b poles.

Now we examine S_1, which is a summation of the tableaux (with sign) of the form

$$\begin{array}{|c|c|c|}\hline \xi & \eta & \zeta \\ \hline\end{array} \qquad \text{(A.2.1)}$$

where $\boxed{\xi}$ and $\boxed{\zeta}$ are rows with total length $m-1$ and they do not involve Q_b. $\boxed{\eta}$ is $\boxed{b}$, $\boxed{b+1}$, $\boxed{\overline{b+1}}$ or $\boxed{\overline{b}}$ for $b \in \{1, 2, \ldots, s-1\}$; $\boxed{s}$, $\boxed{0}$ or $\boxed{\overline{s}}$ for $b = s$. Owing to relations (3.26)–(3.29), S_1 is free of colour b poles under the BAEs (3.2)–(3.5). From now on, we consider S_k for $k \geqslant 2$.

• The case† $b \in \{1, 2, \ldots, s-1\}$: owing to the admissibility conditions, we have only to consider S_2, S_3 and S_4.

S_2 is a summation of the tableaux (with sign) of the form

$$\begin{array}{|c|c|c|c|}\hline \xi & b & b+1 & \eta' \\ \hline\end{array} + \begin{array}{|c|c|c|c|}\hline \xi' & \overline{b+1} & \overline{b} & \zeta \\ \hline\end{array} \qquad \text{(A.2.2)}$$

and

$$\begin{array}{|c|c|c|c|c|}\hline \xi & b & \eta & \overline{b} & \zeta \\ \hline\end{array} + \begin{array}{|c|c|c|c|c|}\hline \xi & b & \eta & \overline{b+1} & \zeta \\ \hline\end{array} + \begin{array}{|c|c|c|c|c|}\hline \xi & b+1 & \eta & \overline{b} & \zeta \\ \hline\end{array}$$
$$+ \begin{array}{|c|c|c|c|c|}\hline \xi & b+1 & \eta & \overline{b+1} & \zeta \\ \hline\end{array}$$
$$= \boxed{\xi}\left(\boxed{b} + \boxed{b+1}\right)\boxed{\eta}\left(\boxed{\overline{b}} + \boxed{\overline{b+1}}\right)\boxed{\zeta} \qquad \text{(A.2.3)}$$

where ($\boxed{\xi}$, $\boxed{\eta'}$), ($\boxed{\xi'}$, $\boxed{\zeta}$) and ($\boxed{\xi}$, $\boxed{\eta}$, $\boxed{\zeta}$) are rows with total length $m-2$, which do not contain $\boxed{b}$, $\boxed{b+1}$, $\boxed{\overline{b+1}}$ and $\boxed{\overline{b}}$. Thus, owing to lemma 3.3 and relations (3.26), (3.29), S_2 does not have colour b poles under the BAE (3.2) and (3.3).

S_3 is a summation of the tableaux (with sign) of the form

$$\begin{array}{|c|c|c|c|c|c|}\hline \xi & b & b+1 & \eta & \overline{b} & \zeta \\ \hline\end{array} + \begin{array}{|c|c|c|c|c|c|}\hline \xi & b & b+1 & \eta & \overline{b+1} & \zeta \\ \hline\end{array}$$
$$= \begin{array}{|c|c|c|c|}\hline \xi & b & b+1 & \eta \\ \hline\end{array}\left(\boxed{\overline{b}} + \boxed{\overline{b+1}}\right)\boxed{\zeta} \qquad \text{(A.2.4)}$$

and

$$\begin{array}{|c|c|c|c|c|c|}\hline \xi & b & \eta' & \overline{b+1} & \overline{b} & \zeta \\ \hline\end{array} + \begin{array}{|c|c|c|c|c|c|}\hline \xi & b+1 & \eta' & \overline{b+1} & \overline{b} & \zeta \\ \hline\end{array}$$
$$= \boxed{\xi}\left(\boxed{b} + \boxed{b+1}\right)\begin{array}{|c|c|c|c|}\hline \eta' & \overline{b+1} & \overline{b} & \zeta \\ \hline\end{array} \qquad \text{(A.2.5)}$$

where ($\boxed{\xi}$, $\boxed{\eta}$, $\boxed{\zeta}$) and ($\boxed{\xi}$, $\boxed{\eta'}$, $\boxed{\zeta}$) are rows with total length $m-3$, which do not contain $\boxed{b}$, $\boxed{b+1}$, $\boxed{\overline{b+1}}$ and $\boxed{\overline{b}}$. Thus, owing to lemma 3.3 and relations (3.26), (3.29), S_3 does not have colour b poles under the BAE (3.2) and (3.3).

S_4 is a summation of the tableaux (with sign) of the form

$$\begin{array}{|c|c|c|c|c|c|c|}\hline \xi & b & b+1 & \eta & \overline{b+1} & \overline{b} & \zeta \\ \hline\end{array} \qquad \text{(A.2.6)}$$

where $\boxed{\xi}$, $\boxed{\eta}$ and $\boxed{\zeta}$ are rows with total length $m-4$, which do not contain $\boxed{b}$, $\boxed{b+1}$, $\boxed{\overline{b+1}}$ and $\boxed{\overline{b}}$. Thus, owing to lemma 3.3, S_4 does not have colour b poles.

† This is void for the $B(0|1)$ case.

• The case $b = s$: Owing to the admissibility conditions, we have only to consider S_2 and S_3.

S_2 is a summation of the tableaux (with sign) of the form

$$\begin{array}{|c|c|c|c|}\hline \xi & s & 0 & \eta \\ \hline\end{array} = \frac{Q_{s-1}(v-s-1)Q_s(v-s+3)}{Q_{s-1}(v-s+1)Q_s(v-s+1)} \times \boxed{\xi} \times \boxed{\eta} \tag{A.2.7}$$

$$\begin{array}{|c|c|c|c|}\hline \xi & s & \overline{s} & \eta \\ \hline\end{array} = \frac{Q_{s-1}(v-s-1)Q_s(v-s+2)}{Q_{s-1}(v-s+1)Q_s(v-s)} \times \frac{Q_{s-1}(v-s+2)Q_s(v-s-1)}{Q_{s-1}(v-s)Q_s(v-s+1)} \times \boxed{\xi} \times \boxed{\eta} \tag{A.2.8}$$

$$\begin{array}{|c|c|c|c|}\hline \xi & 0 & \overline{s} & \eta \\ \hline\end{array} = \frac{Q_{s-1}(v-s+2)Q_s(v-s-2)}{Q_{s-1}(v-s)Q_s(v-s)} \times \boxed{\xi} \times \boxed{\eta} \tag{A.2.9}$$

where $\boxed{\xi}$ and $\boxed{\eta}$ are rows with total length $m-2$, which do not contain $\boxed{s}$, $\boxed{0}$ and $\boxed{\overline{s}}$; $v = u+h$: h is a shift parameter. The colour s residues at $u = u_k^{(s)} + s - 1 - h$ of the functions $\begin{array}{|c|c|c|c|}\hline \xi & s & 0 & \eta \\ \hline\end{array}$ and $\begin{array}{|c|c|c|c|}\hline \xi & s & \overline{s} & \eta \\ \hline\end{array}$ cancel each other out under the BAE (3.4) or (3.5). The colour s residues at $u = u_k^{(s)} + s - h$ of the functions $\begin{array}{|c|c|c|c|}\hline \xi & s & \overline{s} & \eta \\ \hline\end{array}$ and $\begin{array}{|c|c|c|c|}\hline \xi & 0 & \overline{s} & \eta \\ \hline\end{array}$ cancel each other out under the BAE (3.4) or (3.5). Thus, S_2 does not have colour s poles under the BAE (3.4) or (3.5).

S_3 is a summation of the tableaux (with sign) of the form

$$\begin{array}{|c|c|c|c|c|}\hline \xi & s & 0 & \overline{s} & \eta \\ \hline\end{array} \tag{A.2.10}$$

where $\boxed{\xi}$ and $\boxed{\eta}$ are rows with total length $m-3$, which do not contain $\boxed{s}$, $\boxed{0}$ and $\boxed{\overline{s}}$. Thus, owing to lemma 3.3, S_3 does not have colour s poles.

Then $\mathcal{T}_m(u)$ is free of poles under the condition that the BAEs (3.2)–(3.5) are valid; owing to the relation (3.40), this also holds true for $\mathcal{T}_{\lambda\subset\mu}(u)$. In particular, the pole-freeness of $\mathcal{T}^a(u)$ follows immediately.

Appendix A.3. Outline of the proof of theorem 3.1: the $D(r|s)$ ($r \in \mathbb{Z}_{\geqslant 2}$, $s \in \mathbb{Z}_{\geqslant 1}$) case

We prove that $\mathcal{T}^a(u)$ is free of colour b poles, that is, $\mathrm{Res}_{u=u_k^{(b)}+\cdots}\mathcal{T}^a(u) = 0$ for any $b \in \{1, 2, \ldots, s+r\}$ under the condition that the BAE (3.1) is valid. The function $\boxed{c}_u$ (3.14) with $c \in J$ has colour b poles only for $c = b, b+1, \overline{b+1}$ or $\overline{b}$ if $b \in \{1, 2, \ldots, s+r-1\}$; for $c = s+r-1, s+r, \overline{s+r}$ or $\overline{s+r-1}$ if $b = s+r$, so we shall trace only $\boxed{b}$, $\boxed{b+1}$, $\boxed{\overline{b+1}}$ or $\boxed{\overline{b}}$ for $b \in \{1, 2, \ldots, s+r-1\}$; $\boxed{s+r-1}$, $\boxed{s+r}$, $\boxed{\overline{s+r}}$ or $\boxed{\overline{s+r-1}}$ for $b = s+r$. Let S_k be the partial sum of $\mathcal{T}^a(u)$, which contains k boxes among $\boxed{b}$, $\boxed{b+1}$, $\boxed{\overline{b+1}}$ or $\boxed{\overline{b}}$ for $b \in \{1, 2, \ldots, s+r-1\}$; $\boxed{s+r-1}$, $\boxed{s+r}$, $\boxed{\overline{s+r}}$ or $\boxed{\overline{s+r-1}}$ for $b = s+r$. Apparently, S_0 does not have colour b poles.

Next we consider S_1, which is a summation of the tableaux (with sign) of the form

$$\begin{array}{|c|}\hline \xi \\ \hline \eta \\ \hline \zeta \\ \hline\end{array} \tag{A.3.1}$$

where $\boxed{\xi}$ and $\boxed{\zeta}$ are columns with total length $a-1$ and they do not contain Q_b. $\boxed{\eta}$ is $\boxed{b}$, $\boxed{b+1}$, $\boxed{\overline{b+1}}$ or $\boxed{\overline{b}}$ for $b \in \{1, 2, \ldots, s+r-1\}$; $\boxed{s+r-1}$, $\boxed{s+r}$, $\boxed{\overline{s+r}}$ or $\boxed{\overline{s+r-1}}$ for $b = s+r$. Owing to the relations (3.30)–(3.37), S_1 is free of colour b poles under the BAE (3.1). From now on we consider S_k for $k \geqslant 2$.

• The case $b \in \{1, 2, \ldots, s + r - 2\}$: the proof is similar to $B(r|s)$ $(r \in \mathbb{Z}_{\geqslant 1})$ case, so we omit it.

• The case $b = s + r - 1$ or $b = s + r$: S_{2n} $(k = 2n,\ n \in \mathbb{Z}_{\geqslant 2}$†) is a summation of the tableaux (with signs) of the form

$$
\begin{array}{|c|}\hline \xi \\ \hline s+r-1 \\ \hline \overline{s+r} \\ \hline s+r \\ \hline \vdots \\ \hline \overline{s+r} \\ \hline s+r \\ \hline \overline{s+r-1} \\ \hline \zeta \\ \hline \end{array}
+
\begin{array}{|c|}\hline \xi \\ \hline s+r-1 \\ \hline \overline{s+r} \\ \hline s+r \\ \hline \vdots \\ \hline \overline{s+r} \\ \hline s+r \\ \hline \overline{s+r} \\ \hline \zeta \\ \hline \end{array}
+
\begin{array}{|c|}\hline \xi \\ \hline s+r \\ \hline \overline{s+r} \\ \hline s+r \\ \hline \vdots \\ \hline \overline{s+r} \\ \hline s+r \\ \hline \overline{s+r-1} \\ \hline \zeta \\ \hline \end{array}
+
\begin{array}{|c|l}\cline{1-1} \xi & \\ \cline{1-1} s+r & {\scriptstyle v} \\ \cline{1-1} \overline{s+r} & {\scriptstyle v-2} \\ \cline{1-1} s+r & {\scriptstyle v-4} \\ \cline{1-1} \vdots & \\ \cline{1-1} \overline{s+r} & {\scriptstyle v-4n+6} \\ \cline{1-1} s+r & {\scriptstyle v-4n+4} \\ \cline{1-1} \overline{s+r} & {\scriptstyle v-4n+2} \\ \cline{1-1} \zeta & \\ \cline{1-1} \end{array}
= A(v)B(v) \times \boxed{\xi} \times \boxed{\zeta} \qquad \text{(A.3.2)}
$$

and

$$
\begin{array}{|c|}\hline \xi \\ \hline \overline{s+r} \\ \hline s+r \\ \hline \overline{s+r} \\ \hline s+r \\ \hline \vdots \\ \hline \overline{s+r} \\ \hline s+r \\ \hline \overline{s+r} \\ \hline s+r \\ \hline \zeta \\ \hline \end{array}
+
\begin{array}{|c|}\hline \xi \\ \hline s+r-1 \\ \hline s+r \\ \hline \overline{s+r} \\ \hline s+r \\ \hline \vdots \\ \hline \overline{s+r} \\ \hline s+r \\ \hline \overline{s+r} \\ \hline s+r \\ \hline \zeta \\ \hline \end{array}
+
\begin{array}{|c|}\hline \xi \\ \hline \overline{s+r} \\ \hline s+r \\ \hline \overline{s+r} \\ \hline s+r \\ \hline \vdots \\ \hline \overline{s+r} \\ \hline s+r \\ \hline \overline{s+r} \\ \hline \overline{s+r-1} \\ \hline \zeta \\ \hline \end{array}
+
\begin{array}{|c|l}\cline{1-1} \xi & \\ \cline{1-1} s+r-1 & {\scriptstyle v} \\ \cline{1-1} s+r & {\scriptstyle v-2} \\ \cline{1-1} \overline{s+r} & {\scriptstyle v-4} \\ \cline{1-1} s+r & {\scriptstyle v-6} \\ \cline{1-1} \vdots & \\ \cline{1-1} \overline{s+r} & {\scriptstyle v-4n+8} \\ \cline{1-1} s+r & {\scriptstyle v-4n+6} \\ \cline{1-1} \overline{s+r} & {\scriptstyle v-4n+4} \\ \cline{1-1} \overline{s+r-1} & {\scriptstyle v-4n+2} \\ \cline{1-1} \zeta & \\ \cline{1-1} \end{array}
= C(v)D(v) \times \boxed{\xi} \times \boxed{\zeta} \qquad \text{(A.3.3)}
$$

where $v = u + h_1$: h_1 is some shift parameter; $\boxed{\xi}$ and $\boxed{\zeta}$ are columns with total length $a - 2n$, which do not contain $\boxed{s+r-1}$, $\boxed{s+r}$, $\boxed{\overline{s+r}}$ and $\boxed{\overline{s+r-1}}$;

$$
\begin{aligned}
A(v) &= \frac{Q_{s+r-1}(v-s+r+1)}{Q_{s+r-1}(v-s+r-1)} + \frac{Q_{s+r-2}(v-s+r)Q_{s+r-1}(v-s+r-3)}{Q_{s+r-2}(v-s+r-2)Q_{s+r-1}(v-s+r-1)} \\
B(v) &= \frac{Q_{s+r-1}(v-s+r-4n-1)}{Q_{s+r-1}(v-s+r-4n+1)} \\
&\qquad + \frac{Q_{s+r-2}(v-s+r-4n)Q_{s+r-1}(v-s+r-4n+3)}{Q_{s+r-2}(v-s+r-4n+2)Q_{s+r-1}(v-s+r-4n+1)} \\
C(v) &= \frac{Q_{s+r}(v-s+r+1)}{Q_{s+r}(v-s+r-1)} + \frac{Q_{s+r-2}(v-s+r)Q_{s+r}(v-s+r-3)}{Q_{s+r-2}(v-s+r-2)Q_{s+r}(v-s+r-1)} \\
D(v) &= \frac{Q_{s+r}(v-s+r-4n-1)}{Q_{s+r}(v-s+r-4n+1)} + \frac{Q_{s+r-2}(v-s+r-4n)Q_{s+r}(v-s+r-4n+3)}{Q_{s+r-2}(v-s+r-4n+2)Q_{s+r}(v-s+r-4n+1)}.
\end{aligned}
\qquad \text{(A.3.4)}
$$

† The case $n = 1$ can be treated similarly.

Apparently, $A(v)$ and $B(v)$ (resp. $C(v)$ and $D(v)$) do not contain Q_{s+r} (resp. Q_{s+r-1}). One can also check $A(v)$ and $B(v)$ (resp. $C(v)$ and $D(v)$) are free of colour $s+r-1$ (resp. $s+r$) poles under the BAE (3.1).

S_{2n+1} ($k = 2n+1$, $n \in \mathbb{Z}_{\geqslant 2}$†) is a summation of the tableaux (with signs) of the form

$$
\begin{array}{|c|}\hline \xi \\ \hline s+r \\ \hline \overline{s+r} \\ \hline s+r \\ \hline \vdots \\ \hline \overline{s+r} \\ \hline s+r \\ \hline \overline{s+r} \\ \hline s+r \\ \hline \zeta \\ \hline \end{array}
+
\begin{array}{|c|}\hline \xi \\ \hline s+r \\ \hline \overline{s+r} \\ \hline s+r \\ \hline \vdots \\ \hline \overline{s+r} \\ \hline s+r \\ \hline \overline{s+r} \\ \hline \overline{s+r-1} \\ \hline \zeta \\ \hline \end{array}
+
\begin{array}{|c|}\hline \xi \\ \hline s+r-1 \\ \hline \overline{s+r} \\ \hline s+r \\ \hline \vdots \\ \hline \overline{s+r} \\ \hline s+r \\ \hline \overline{s+r} \\ \hline s+r \\ \hline \zeta \\ \hline \end{array}
+
\begin{array}{|c|l}\cline{1-1} \xi & \\ \cline{1-1} s+r-1 & v \\ \cline{1-1} \overline{s+r} & v-2 \\ \cline{1-1} s+r & v-4 \\ \cline{1-1} \vdots & \\ \cline{1-1} \overline{s+r} & v-4n+6 \\ \cline{1-1} s+r & v-4n+4 \\ \cline{1-1} \overline{s+r} & v-4n+2 \\ \cline{1-1} \overline{s+r-1} & v-4n \\ \cline{1-1} \zeta & \\ \cline{1-1} \end{array}
= E(v)F(v) \times \boxed{\xi} \times \boxed{\zeta} \qquad \text{(A.3.5)}
$$

and

$$
\begin{array}{|c|}\hline \xi \\ \hline \overline{s+r} \\ \hline s+r \\ \hline \overline{s+r} \\ \hline s+r \\ \hline \vdots \\ \hline \overline{s+r} \\ \hline s+r \\ \hline \overline{s+r} \\ \hline \zeta \\ \hline \end{array}
+
\begin{array}{|c|}\hline \xi \\ \hline s+r-1 \\ \hline s+r \\ \hline \overline{s+r} \\ \hline s+r \\ \hline \vdots \\ \hline \overline{s+r} \\ \hline s+r \\ \hline \overline{s+r} \\ \hline \zeta \\ \hline \end{array}
+
\begin{array}{|c|}\hline \xi \\ \hline \overline{s+r} \\ \hline s+r \\ \hline \overline{s+r} \\ \hline s+r \\ \hline \vdots \\ \hline \overline{s+r} \\ \hline s+r \\ \hline \overline{s+r-1} \\ \hline \zeta \\ \hline \end{array}
+
\begin{array}{|c|l}\cline{1-1} \xi & \\ \cline{1-1} s+r-1 & v \\ \cline{1-1} s+r & v-2 \\ \cline{1-1} \overline{s+r} & v-4 \\ \cline{1-1} s+r & v-6 \\ \cline{1-1} \vdots & \\ \cline{1-1} \overline{s+r} & v-4n+4 \\ \cline{1-1} s+r & v-4n+2 \\ \cline{1-1} \overline{s+r-1} & v-4n \\ \cline{1-1} \zeta & \\ \cline{1-1} \end{array}
= G(v)H(v) \times \boxed{\xi} \times \boxed{\zeta} \qquad \text{(A.3.6)}
$$

where $v = u + h_2$: h_2 is some shift parameter; $\boxed{\xi}$ and $\boxed{\zeta}$ are columns with total length $a-2n-1$, which do not contain $\boxed{s+r-1}$, $\boxed{s+r}$, $\boxed{\overline{s+r}}$ and $\boxed{\overline{s+r-1}}$;

$$
\begin{aligned}
E(v) &= \frac{Q_{s+r-1}(v-s+r+1)}{Q_{s+r-1}(v-s+r-1)} + \frac{Q_{s+r-2}(v-s+r)Q_{s+r-1}(v-s+r-3)}{Q_{s+r-2}(v-s+r-2)Q_{s+r-1}(v-s+r-1)} \\
F(v) &= \frac{Q_{s+r}(v-s+r-4n-3)}{Q_{s+r}(v-s+r-4n-1)} + \frac{Q_{s+r-2}(v-s+r-4n-2)Q_{s+r}(v-s+r-4n+1)}{Q_{s+r-2}(v-s+r-4n)Q_{s+r}(v-s+r-4n-1)} \\
G(v) &= \frac{Q_{s+r}(v-s+r+1)}{Q_{s+r}(v-s+r-1)} + \frac{Q_{s+r-2}(v-s+r)Q_{s+r}(v-s+r-3)}{Q_{s+r-2}(v-s+r-2)Q_{s+r}(v-s+r-1)} \qquad \text{(A.3.7)} \\
H(v) &= \frac{Q_{s+r-1}(v-s+r-4n-3)}{Q_{s+r-1}(v-s+r-4n-1)} \\
&\quad + \frac{Q_{s+r-2}(v-s+r-4n-2)Q_{s+r-1}(v-s+r-4n+1)}{Q_{s+r-2}(v-s+r-4n)Q_{s+r-1}(v-s+r-4n-1)}.
\end{aligned}
$$

† The case $n = 1$ can be treated similarly.

Apparently, $E(v)$ and $H(v)$ (resp. $F(v)$ and $G(v)$) do not contain Q_{s+r} (resp. Q_{s+r-1}). One can also check that $E(v)$ and $H(v)$ (resp. $F(v)$ and $G(v)$) are free of colour $s+r-1$ (resp. $s+r$) poles under the BAE (3.1).

Thus, S_k have neither colour $s+r-1$ poles nor colour $s+r$ poles under the BAE (3.1).

Appendix B. Generating series for $\mathcal{T}^a(u)$ and $\mathcal{T}_m(u)$

The functions $\mathcal{T}^a(u)$ and $\mathcal{T}_m(u)$ ($a, m \in \mathbb{Z}$; $u \in \mathbb{C}$) are determined by the following non-commutative generating series.

The $B(r|s)$ case:

$$\begin{aligned}
&(1+\boxed{\overline{1}}X)^{-1}\ldots(1+\boxed{\overline{s}}X)^{-1}(1+\boxed{\overline{s+1}}X)\cdots(1+\boxed{\overline{s+r}}X)(1-\boxed{0}X)^{-1}\\
&\qquad\times(1+\boxed{s+r}X)\ldots(1+\boxed{s+1}X)(1+\boxed{s}X)^{-1}\cdots(1+\boxed{1}X)^{-1}\\
&\qquad=\sum_{a=-\infty}^{\infty}\mathcal{T}^a(u+a-1)X^a
\end{aligned} \tag{B.1}$$

$$\begin{aligned}
&(1-\boxed{1}X)\ldots(1-\boxed{s}X)(1-\boxed{s+1}X)^{-1}\ldots(1-\boxed{s+r}X)^{-1}(1+\boxed{0}X)\\
&\qquad\times(1-\boxed{\overline{s+r}}X)^{-1}\ldots(1-\boxed{\overline{s+1}}X)^{-1}(1-\boxed{\overline{s}}X)\cdots(1-\boxed{\overline{1}}X)\\
&\qquad=\sum_{m=-\infty}^{\infty}\mathcal{T}_m(u+m-1)X^m.
\end{aligned} \tag{B.2}$$

The $D(r|s)$ case:

$$\begin{aligned}
&(1+\boxed{\overline{1}}X)^{-1}\ldots(1+\boxed{\overline{s}}X)^{-1}(1+\boxed{\overline{s+1}}X)\ldots(1+\boxed{\overline{s+r}}X)(1-\boxed{s+r}X\boxed{\overline{s+r}}X)^{-1}\\
&\qquad\times(1+\boxed{s+r}X)\cdots(1+\boxed{s+1}X)(1+\boxed{s}X)^{-1}\cdots(1+\boxed{1}X)^{-1}\\
&\qquad=\sum_{a=-\infty}^{\infty}\mathcal{T}^a(u+a-1)X^a
\end{aligned} \tag{B.3}$$

$$\begin{aligned}
&(1-\boxed{1}X)\ldots(1-\boxed{s}X)(1-\boxed{s+1}X)^{-1}\ldots(1-\boxed{s+r-1}X)^{-1}\\
&\qquad\times[(1-\boxed{s+r}X)^{-1}+(1-\boxed{\overline{s+r}}X)^{-1}-1]\\
&\qquad\times(1-\boxed{\overline{s+r-1}}X)^{-1}\ldots(1-\boxed{\overline{s+1}}X)^{-1}(1-\boxed{\overline{s}}X)\ldots(1-\boxed{\overline{1}}X)\\
&\qquad=\sum_{m=-\infty}^{\infty}\mathcal{T}_m(u+m-1)X^m.
\end{aligned} \tag{B.4}$$

Here X is a shift operator $X = \mathrm{e}^{2\partial_u}$. In particular, we have $\mathcal{T}^0(u) = 1$; $\mathcal{T}_0(u) = 1$; $\mathcal{T}^a(u) = 0$ for $a < 0$; $\mathcal{T}_m(u) = 0$ for $m < 0$.

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